ESCIM 2025

BOOK OF ABSTRACTS

16th European Symposium on Computational Intelligence and Mathematics

May 18th – 21st, 2025 • A Coruña, Spain

Editors:

László T. Kóczy, Jesús Medina

Associate Editors:

Roberto G. Aragón, Fernando Chacón-Gómez, Janusz Kacprzyk,

M. Eugenia Cornejo-Piñero, David Lobo, Francisco J. Ocaña-Alcázar, Eloísa Ramírez-Poussa

Book of abstracts of ESCIM 2025

Edition 1st
First published: 2025

ISBN: 978-84-09-73668-3
Published and printed by:
Universidad de Cádiz (Dept. Matemáticas), Spain.

Table of Contents

Program Committee

María José Benítez-Caballero	University of Cádiz. Spain
Pedro Cabalar	Univ. A Coruña. Spain
João Paulo Carvalho	University of Lisbon. Portugal
Oscar Castillo	Tijuana Institute of Technology. México
Agata Ciabattoni	TU Wien. Austria
Davide Ciucci	University of Milano-Bicocca. Italy
Stefania Costantini	University of L'Aquila. Italy
Miroslav Ćirić	University of Niš. Serbia
M. Eugenia Cornejo	University of Cádiz, Spain
Chris Cornelis	University of Gante. Belgium
Bernard De Baets	University of Gante. Belgium
Christian G. Fermüller	TU Wien. Austria
Péter Földesi	Univ. Széchenyi István. Hungary
Robert Fullér	Obuda University, Hungary
Lluís Godo	Artificial Intelligence Research Institute, IIIA-CSIC. Spain
István Á. Harmati	Univ. Széchenyi István. Hungary
Piotr A. Kowalski	AGH Univ. of Science and Technology. Poland
Stanislav Krajči	UPJS Košice. Slovakia
Ondrej Krídlo	UPJS Košice. Slovakia
David Lobo	University of Cádiz. Spain
Nicolás Madrid	University of Cádiz. Spain
Juan Moreno García	University of Castilla La Mancha. Spain
Szilvia Nagy	Univ. Széchenyi István. Hungary
Manuel Ojeda-Aciego	University of Málaga. Spain
Irina Perfilieva	Centre of Excellence IT4Innovations. Univ. Ostrava. Czech Republic
Jozef Pócs	Slovak Academy of Sciences. Slovakia
Claudiu Pozna	Transilvania University of Brasov. Romania
Radu-Emil Precup	Politehnica University of Timisoara. Rumania
Marek Reformat	University of Alberta. Canada
Eloísa Ramírez-Poussa	University of Cádiz. Spain
Alexander Šostaks	University of Latvia. Latvia
Andreja Tepavčević	University of Novi Sad. Serbia
Esko Turunen	Tampere University of Technology. Finland

Program of the

16th European Symposium on Computational Intelligence and Mathematics including Workshop DigForASP

May 18th - 21st, 2025. A Coruña, Spain

MONDAY 19th Location: EXE Eurostarts Hotel	
8:30	Open Registration Desk
9:00–9:30	Inauguration
9:30–10:20	**Keynote Speaker - Vladik Kreinovich** Title: Spectacular Successes of Deep Learning: Why and What Next? Chairperson: Jesús Medina
10:20–11:20	**Regular Session S1** Chairperson: Vladik Kreinovich
	Virtual Assistant supported by Fusion of Fuzzy Rule-based Explainers Pablo Pérez, Alejandro Catalá, Alberto Bugarín, José Alonso
	Fuzzy Weighted Shapley Values based on Nearest Neighbor Classification Marcin Ostrowski, Katarzyna Kaczmarek-Majer, Jan Budzianowski, Paweł Burchardt, Janusz Rzeźniczak
	Logarithmic Number System Is Optimal for AI Computations: Theoretical Explanation of Empirical Success Olga Kosheleva, Vladik Kreinovich, Christoph Lauter, Kristalys Ruiz-Rohena
11:20–11:40	Coffee break
11:40–13:20	**Special Session: Homage to Alexander Šostak** Chairperson: Olga Grigorenko
	Relational Characterization of b-Metrics Olga Grigorenko
	On Application of Formal Fuzzy Contexts in Data Comparison Models Maris Krastins
	Solving linear equations using generalized equality Branimir Šešelja, Andreja Tepavčević
	Degree of validity of attribute implications based on multi-adjoint concept-forming M. Eugenia Cornejo, Jesús Medina, Francisco J. Ocaña-Alcázar
	(Ab) Comparison of Formal Concept Analysis and k-means Clustering on Healthcare Dataset Štefan Puci, Matúš Semančík, Dominika Kotlárová, L'ubomír Antoni, Peter Eliaš, Ján Guniš, Stanislav Krajči, Ondrej Krídlo, L'ubomir Šnajder
13:20–14:40	Lunch

14:40–15:30	**Keynote Speaker - Marek Reformat** Title: Category Theory in Action: A Novel Framework for Knowledge Graph Analysis and Reasoning. Chairperson: László T. Kóczy.
15:30–16:00	Coffee break
16:00–17:20	**Regular Session S2** Chairperson: Marek Reformat
	Decision-making based on confirmation measures Fernando Chacón-Gómez, M. Eugenia Cornejo, Jesús Medina
	Network Visualization of a Fuzzy Measure Carlos I. Pérez-Sechi, Javier Castro, Daniel Gómez, Inmaculada Gutiérrez, Daniel Martín, Rosa Espínola
	Complex-Valued Neural Networks: Applications and Performance in Image Classification Piotr A. Kowalski, Michał Rutkowski
	The granular structure of covering-based rough sets generated by the Mapper algorithm Wannes De Maeyer, Mauricio Restrepo, Chris Cornelis
17:20–18:15	**Regular Session S3** Chairperson: Radko Mesiar
	Problem of distributivity between 2-uninorms and quasy-linear means Dragan Jočić
	Choquet-inspired functions Humberto Bustince, Radko Mesiar, Gracaliz Dimuro, Bernard De Baets
	(Ab) Unifunctions, a new class of functions for bipolar modeling Juraj Kalafut, Martin Kalina
20:00	**Welcome reception**

TUESDAY 20th **Location: EXE Eurostarts Hotel**	
9:00–9:50	**Keynote Speaker - Pedro Cabalar** Title: Normative Reasoning with Deontic Answer Set Programming. Chairperson: Stefania Costantini
9:50–10:50	**Workshop DigForASP. Session 1** Chairperson: Pedro Cabalar
	Formal Verification of Social Engineering in Information-Seeking Dialogues Andreas Brännström, Juan Carlos Nieves
	Blueprint Personas in Digital Forensics and Cybersecurity Alina Vozna, Stefania Costantini, Andrea Monaldini, Raffaele Olivieri
	(Ab) SM-based Semantics for Answer Set Programs Containing Conditional Literals and Arithmetic Yuliya Lierler
10:50–11:10	Coffee break
11:10–12:00	**Workshop DigForASP. Session 2** Chairperson: Juan Carlos Nieves
	(Ab) Recursive Aggregates as Intensional Functions in Answer Set Programmin Jorge Fandinno, Zachary Hansen
	(Ab) Temporal Rules for Fraud Detection Przemyslaw Walega
	(Ab) Unfiltered Large Language Models: Forensic and Legal Challenges of their Criminal Use on the Dark Web. Preliminary Study Dévika Pérez-Medina, Manuel Ojeda-Hernández, Fernando Chacón-Gómez
12:00–13:30	**Special Session: Decision and optimization model applied to logistics and transport** Chairperson: Julio Alberto López-Gómez
	Time-Dependent Triangular Fuzzy Soft Set Model for Describing Traffic Dynamics Boldizsár Tüű-Szabó, Ruba AlMahasneh, László T. Kóczy
	Learning label ranking trees using evolutionary algorithms Arturo Fernández-Mora, José Antonio Gámez, Juan Carlos Alfaro, Juan Moreno-García
	(Ab) Fuzzy logic applied to decision and optimization models David Muñoz Valero, Juan Moreno-García, Julio Alberto López-Gómez, Enrique Adrián Villarrubia-Martín
	A multi-adjoint lattice logic based on Gödel logic M. Eugenia Cornejo, Francesc Esteva, Luis Fariñas del Cerro, Lluis Godo, Jesús Medina
	Relations Among Intermediate Syllogisms Petra Murinová, Vilém Novak
13:30–15:00	Lunch
15:00–16:15	**Regular Session S4** Chairperson: Manuel Ojeda-Aciego
	On testing hypotheses in fuzzy surroundings Aleksandar Takači, Ivana Štajner-Papuga, Zagorka Lozanov-Crvenković, Gabrijela Grujić, Tatjana Došenović, Dragan Jočić
	Kolmogorov Representation Theorem Explained in Terms of Fuzzy Partitions Irina Perfilieva
	Systems of fuzzy relational equations in case of a complete codomain lattice Vanja Stepanović, Andreja Tepavčević
	(Ab) WAMs based on a pair of n-tuples (w, b) and their relationship to OWA and IOWA operators Andrea Stupňanová, Radko Mesiar, Radomír Halaš, Jozef Pócz
17:00–20:30	**A Coruña tour**
20:30	**Gala dinner**

WEDNESDAY 21st Location: EXE Eurostarts Hotel	
10:00–10:50	**Keynote Speaker - Bernadette Bouchon-Meunier** Title: Entropies, measures of uncertain and imprecise information. Chairperson: László T. Kóczy.
10:50–11:10	Coffee break
11:10–12:00	**Regular Session S5** Chairperson: Bernadette Bouchon-Meunier
	Analytical Properties of General Unary Hypotheses Automaton Quantifiers Martina Daňková
	(Ab) Linear spline copulas Radko Mesiar, Andrea Stupňanová, Fateme Kouchakinejad
	(Ab) Some notes on permutation-equivariant decomposition integrals Adam Šeliga
12:00–13:40	**Special Session: Recent trends in knowledge representation and modelling** Chairperson: Roberto G. Aragón
	Vertical sum of concept lattices via bonds Roberto G. Aragón, Jesús Medina, Samuel Molina
	Bipolar max-product fuzzy relation equations with floor negation Antonio Castejón, David Lobo, Pablo López-Molina
	(Ab) New Simplification Rules for Databases with Positive and Negative Attributes Domingo López-Rodríguez, Manuel Ojeda-Hernández, Carlos Bejines
	(Ab) Formal concept analysis and its fuzzy extensions for Light-Bot game solutions Dominika Kotlárová, Peter Eliaš, Ján Guniš, L'ubomir Antoni, Stanislav Krajči, Ondrej Krídlo, L'ubomír Šnajder
	(Ab) Composition as a fuzzy conjunction between indexes of inclusion Nicolás Madrid, Manuel Ojeda-Aciego
	(Ab) Towards the use of the f-index of inclusion for fuzzy reasoning Nicolás Madrid, Eloísa Ramírez-Poussa, Carolina Díaz-Montarroso
13:40–15:00	**Closing Session**

Social Events	
SUNDAY 18th	
08:30	**Tour to Santiago de Compostela**
MONDAY 19th	
20:00	**Welcome reception**
TUESDAY 20th	
17:00–20:30	**A Coruña tour**
20:30	**Gala dinner**
WEDNESDAY 21st	
13:40–15:00	**Closing Session**

***Keynote speech*:**

Spectacular Successes of Deep Learning: Why and What Next?

Vladik Kreinovich

Department of Computer Science, University of Texas,
El Paso, United States

Abstract: Successes of deep learning are partly due to appropriate selection of activation function, pooling functions, etc. Most of these choices have been made based on empirical comparison and heuristic ideas. In the first – why — part of this talk, we show that many of these choices – and the surprising success of deep learning in the first place – can be explained by reasonably simple and natural mathematics.

In the second – what next — part, we present ideas on how to overcome limitations of current deep learning techniques. One of these limitations is that, in contrast to humans who can learn from a few examples and learn fast, modern deep learning techniques require a large amount of data to learn, and they take a long time to train. In this talk, we show that neural networks do have a potential to learn from a small number of examples – and learn fast. We speculate that the corresponding idea may already be implicitly implemented in Large Language Models – which may partially explain their (somewhat mysterious) success.

Virtual Assistant supported by Fusion of Fuzzy Rule-based Explainers

Pablo Miguel Pérez-Ferreiro, Alejandro Catala, Alberto Bugarín-Diz and Jose M. Alonso-Mora

Centro Singular de Investigacion en Tecnoloxías Intelixentes (CiTIUS),
Universidade de Santiago de Compostela, Spain.
e-mail: pablomiguel.perez@rai.usc.es, alejandro.catala@usc.es,
alberto.bugarin.diz@usc.es, josemaria.alonso.moral@usc.es

Abstract: In this work, we introduce a novel methodology for combining different explanatory inputs into a singular coherent narrative in the medical domain. As proof of concept, we have implemented a prototype ready to diagnose diabetes and heart disease. The proposal is validated with a user study in which we collected 48 valid responses to a given questionnaire. We compared four different strategies for explanation fusion, considering explanations coming from two different fuzzy rule bases. The winner strategy (i.e., verbalizing common antecedents as the main factors of the diagnosis and adding any non-common antecedents as supporting evidence) was perceived as trustworthy, useful, and natural by most participants in the study.

Keywords: Explainable Fuzzy Systems · Natural Language Generation · Trustworthy Conversational Agents · Human Evaluation.

Acknowledgement: All authors acknowledge the support of the Galician Ministry of Culture, Education, Professional Training and University (grants ED431G2023/04 and ED431C2022/19) and the European Union (European Regional Development Fund - ERDF). This work is also supported by the Spanish Ministry of Science and Innovation (MCIN/AEI/10.13039/501100011033/) with grants PID2021-123152OB-C21, PID2020-112623GB-I00, and TED2021-130295B-C33 (this also funded by the "European Union NextGenerationEU/PRTR".

References

1. Ali, S., Abuhmed, T., El-Sappagh, S., Muhammad, K., Alonso-Moral, J.M., Confalonieri, R., Guidotti, R., Ser, J.D., Díaz-Rodríguez, N., Herrera, F.: Explainable artificial intelligence (XAI): What we know and what is left to attain trustworthy artificial intelligence. Information Fusion (2023). https://doi.org/10.1016/j.inffus.2023.101805
2. Alonso, J.M., Castiello, C., Magdalena, L., Mencar, C.: Explainable Fuzzy Systems - Paving the Way from Interpretable Fuzzy Systems to Explainable AI Systems, vol. 970. Springer International Publishing (2021). https://doi.org/10.1007/978-3-030-71098-9

3. Cambria, E., Malandri, L., Mercorio, F., Mezzanzanica, M., Nobani, N.: A survey on xai and natural language explanations. Information Processing and Management **60**(103111), 1–6 (2023). https://doi.org/10.1016/j.ipm.2022.103111
4. Groza, A., Toderean, L., Muntean, G.A., Nicoara, S.D.: Agents that argue and explain classifications of retinal conditions. Journal of Medical and Biological Engineering **41**(5), 730–741 (2021). https://doi.org/10.1007/s40846-021-00647-7
5. Gunning, D., Vorm, E., Wang, J., Turek, M.: DARPA's explainable ai (XAI) program: A retrospective. Applied AI Letters (2021). https://doi.org/10.1002/ail2.61
6. Nauta, M., Trienes, J., Pathak, S., Nguyen, E., Peters, M., Schmitt, Y., Schlötterer, J., van Keulen, M., Seifert, C.: From anecdotal evidence to quantitative evaluation methods: A systematic review on evaluating explainable ai. ACM Computing Surveys (2023). https://doi.org/10.1145/3583558
7. Reiter, E.: Natural Language Generation. Springer Cham (2024). https://doi.org/10.1007/978-3-031-68582-8
8. Soto-Hidalgo, J., Alonso, J., Acampora, G., Alcala-Fdez, J.: JFML: A Java library to design fuzzy logic systems according to the IEEE Std 1855-2016. IEEE Access **6**, 54952–54964 (2018). https://doi.org/10.1109/ACCESS.2018.2872777
9. Stepin, I., Alonso, J.M., Catala, A., Pereira-Fariña, M.: Generation and evaluation of factual and counterfactual explanations for decision trees and fuzzy rule-based classifiers. In: IEEE WCCI (2020). https://doi.org/10.1109/FUZZ48607.2020.9177629
10. Stepin, I., Alonso, J.M., Catala, A., Pereira-Fariña, M.: An empirical study on how humans appreciate automated counterfactual explanations which embrace imprecise information. Information Sciences **618**, 379–399 (2022). https://doi.org/10.1016/j.ins.2022.10.098
11. Think Tank, European Parliament: Artificial intelligence in healthcare: applications, risks, and ethical and social impacts (2022), https://www.europarl.europa.eu/thinktank/en/document/EPRS_STU(2022)729512

Fuzzy Weighted Shapley Values Based on Nearest Neighbor Classification

Marcin Ostrowski[1], Janusz Rzeźniczak[3], Jan Budzianowski[2], Paweł Burchardt[3,4] and Katarzyna Kaczmarek-Majer[1]

[1] Systems Research Institute Polish Academy of Sciences, Newelska 6, 01-147 Warsaw, Poland.

e-mail: Marcin.Ostrowski@ibspan.waw.pl, k.kaczmarek@ibspan.waw.pl

[2] Nowa Sól Multidisciplinary Hospital, 67-100 Nowa Sól, Poland.

[3] Department of Cardiology, J. Struś Hospital, 61-285 Poznań, Poland.

[4] Department of Hypertension, Angiology and Internal Medicine, Poznań University of Medical Sciences, 61-848 Poznań, Poland.

Abstract: In this study, we propose a novel modification of the Exact Shapley Values Based on Nearest Neighbor Classification method that incorporates fuzzy logic to better account for uncertainty in datasets. The modification introduces a fuzzy weight vector computed using the Fuzzy K Nearest Neighbor algorithm, which improves the computation of Shapley values for pseudo-labeled instances. This approach aims to better reflect the relevance of observations and mitigate the effects of data uncertainty. The method was validated on a real-world pilot study of 654 patients diagnosed with acute coronary syndrome. In the dataset used, one of the problems encountered in clinical practice, data uncertainty, emerged. The results show that the K-FVSNN method achieves competitive performance and maintains robust results even with up to 90% missing labels in the training set. These results highlight the potential of the K-FVSNN method for handling uncertain data in medical applications. Future work will explore its application to other datasets and further refinements to the weight vector to improve its generalizability.

Keywords: Shapley Values · semi-supervised learning · classification · missing data · medical data.

Acknowledgement: The project "ExplainMe: Explainable Artificial Intelligence for Monitoring Acoustic Features extracted from Speech" (FENG.02.02-IP.05-0302/23) is carried out within the First Team programme of the Foundation for Polish Science co-financed by the European Union under the European Funds for Smart Economy 2021-2027 (FENG).

References

1. Kursa, M.B., Rudnicki, W.R.: Feature Selection with the Boruta Package. J. Stat. Softw. 36(11), 1-13 (2010). doi:10.18637/jss.v036.i11

2. Chapelle, O., Scholkopf, B., Zien, A.: Semi-Supervised Learning (Chapelle, O. et al., Eds.; 2006) [Book reviews]. IEEE Trans. Neural Netw. 20(3), 542-542 (2009). doi:10.1109/TNN.2009.2015974
3. Lenz, O.U., Peralta, D., Cornelis, C.: Polar Encoding: A Simple Baseline Approach for Classification With Missing Values. IEEE Trans. Fuzzy Syst. 32(5), 3084-3093 (2024). doi:10.1109/TFUZZ.2024.3367419
4. Courtnage, C., Smirnov, E.: Shapley-Value Data Valuation for Semi-supervised Learning. In: Soares, C., Torgo, L. (eds.) Discovery Science, pp. 94-108. Springer, Cham (2021). doi:10.1007/978-3-030-88942-5
5. Ghorbani, A., Zou, J.: Data Shapley: Equitable Valuation of Data for Machine Learning. arXiv preprint (2019). https://arxiv.org/abs/1904.02868
6. Jia, R., Dao, D., Wang, B., Hubis, F.A., Gurel, N.M., Li, B., Zhang, C., Spanos, C.J., Song, D.: Efficient Task-Specific Data Valuation for Nearest Neighbor Algorithms. arXiv preprint (2020). https://arxiv.org/abs/1908.08619
7. Lundberg, S., Lee, S.-I.: A Unified Approach to Interpreting Model Predictions. arXiv preprint (2017). https://arxiv.org/abs/1705.07874
8. Amini, M.-R., Feofanov, V., Pauletto, L., Hadjadj, L., Devijver, E., Maximov, Y.: Self-Training: A Survey. arXiv preprint (2024). https://arxiv.org/abs/2202.12040
9. Shapley, L.S.: A Value for n-Person Games. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games II, pp. 307-317. Princeton University Press, Princeton (1953).
10. Triguero, I., García, S., Herrera, F.: Self-labeled techniques for semi-supervised learning: taxonomy, software and empirical study. Knowl. Inf. Syst. 42, 245-284 (2015). https://api.semanticscholar.org/CorpusID:1955810
11. Yarowsky, D.: Unsupervised Word Sense Disambiguation Rivaling Supervised Methods. In: Proceedings of the Annual Meeting of the Association for Computational Linguistics (1995). https://api.semanticscholar.org/CorpusID:1487550
12. Van Engelen, J.E., Hoos, H.H.: A survey on semi-supervised learning. Mach. Learn. 109, 373-440 (2019). https://api.semanticscholar.org/CorpusID:254738406

Logarithmic Number System Is Optimal for AI Computations: Theoretical Explanation of an Empirical Success

Olga Kosheleva, Vladik Kreinovich, Christoph Lauter and Kristalys Ruiz-Rohena

University of Texas at El Paso, El Paso TX 79968, USA
e-mail: {olgak,vladik,cqlauter}@utep.edu, kruizrohena@miners.utep.edu

Abstract: Current AI systems are very successful, but they are not perfect, they need to be trained better. Training modern AI systems requires a tremendous amount of computations – that already take a lot of time. To increase the number of computations, we need to make each computation step faster. It was shown that we can speed up computations if we apply an appropriate nonlinear transformation to all the values, and that logarithmic transformation leads to the fastest speedup. In this paper, we provide a theoretical explanation for this empirical success.

Keywords: Machine learning · AI · Nonlinear transformations · Logarithmic transformation · Optimal transformation.

Acknowledgement: This work was supported by the National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, EAR-222539, by the AT&T Fellowship in Information Technology, by a grant from the Hungarian National Research, Development and Innovation Office (NRDI), and by the Institute for Risk and Reliability, Leibniz Universitaet Hannover, Germany.

References

1. Christ, M., de Dinechin, F., Pétrot, F.: Low-precision logarithmic arithmetic for neural network accelerators. Proceedings of the 2022 IEEE 33rd International Conference on Application-specific Systems, Architectures, and Processes ASAP 2022, Gothenburg, Sweden, July 12–14, 2022, pp. 72–78.
2. Nguyen, H. T., Kreinovich, V.: Applications of continuous mathematics to computer science. Kluwer, Dordrecht (1997)
3. Niu, Z. et al.: Hardware-efficient logarithmic floating-point multipliers for error-tolerant applications. IEEE Transactions on Circuits and Systems–I **71**(1), 209–222 (2024)
4. Thorne, K. S., Blandford, R. D.: Modern classical physic. Princeton University Press, Princeton, New Jersey (2021)
5. Zhao, J., et al.: LNS-Madam: low-precision training in logarithmic numer system using multiplicative weight upodate. IEEE Transactions on Computers **72**(12), 3179–3190 (2022)

Relational Characterization of b-Metrics

Olga Grigorenko and Reinis Isaks

Department of Mathematics, University of Latvia, 3 Jelgavas Street,
Riga, LV-1004, Latvia
e-mail: olga.grigorenko@lu.lv, ri19002@edu.lu.lv

Abstract: This paper investigates the concept of b-metrics through relational structures. In particular, we examine commonly used definitions of b-metrics and establish a unique characterization of b-metrics in terms of linear order relations. Our findings provide a deeper understanding of the properties of b-metrics and their underlying relational framework.

Keywords: B-Metric · Metric · Fuzzy Metric.

Acknowledgement: This research is funded by the Latvian Council of Science, project "A fuzzy logic based approach to the value of information estimation in optimal control problems under uncertainty with applications to ecological management", project No. lzp-2024/1-0188.

References

1. Šostak, A.: Some remarks on fuzzy k-pseudometric spaces, Filomat **32:10**, 3567-3580 (2018).
2. Ahmad, M. A., Ahmad, A., Esi, A.: Automated assessment and optimization of fuzzy logic in metric space applications. Journal of Science and Arts **24**(2), 295-304 (2024).
3. Atallaoui, S., Tallafha, A., Shatanawi, W.: Some Fixed Point Results on Extended b-Metric Spaces by Utilizing C-Class Functions, International Journal of Applied Mathematics **33**(6), (2020).
4. Bakhtin, I.A.: The contraction mapping principle in almost metric spaces, Functional Analysis **30**, 26-37 (1989).
5. Berinde, V., Păcurar, M.: The early developments in fixed point theory on b-metric spaces: a brief survey and some important related aspects, Carpathian Journal of Mathematics Vol **38**(3), 523-538 (2022).
6. Bourbaki, N.: Topologie Generale, Herman, Paris, France (1974).
7. Cobzaş, S., Czerwik, S.: The Completion of Generalized B-Metric Spaces and Fixed Points, Fixed Point Theory **21**(1), 133-150 (2020).
8. Czerwik, S.: Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostravensis **1**, 5-11 (1993).
9. Fréchet, M.: Sur quelques points du calcul fonctionnel, Thèse, Paris, 1905. Rendiconti Circolo Mat. Palermo **22**, 1-74 (1906).
10. George, A., Veeramani, P.: On Some Results in Fuzzy Metric Spaces, Fuzzy Sets Syst. **64**, 395-399 (1994).
11. Gregori, V., Morillas, S., Sapena, A.: Examples of fuzzy metrics and applications. Fuzzy Sets and Systems **170**(1), 95-111 (2011).

12. George, R., Radenović, S., Reshma, K. P., Shukla, S.: Rectangular b-Metric Space and Contraction Principles, J. Nonlinear Sci. Appl. **8**, 1005-1013 (2015).
13. Grigorenko, O., Šostak, A.: Fuzzy metrics in terms of fuzzy relations, Math. **11**(16), 3528 (2023).
14. Kamran, T., Samreen, M., Ain, Q. U.: A Generalization of b-Metric Space and Some Fixed Point Theorems, Mathematics **5**(1), 10 (2017).
15. Khemaratchatakumthorn, T., Pongsriiam, P., Samphavat, S.: Further remarks on b-metrics, metric-preserving functions, and other related metrics. International Journal of Mathematics and Computer Science **14**(2), 473-480 (2019).
16. Kramosil, I., Michalek, J.: Fuzzy Metrics and Statistical Metric Spaces, Kybernetika **11**, 336-344 (1975).
17. Künzi, H., Pajoohesh, H.: b-metrics. Topology and its Applications **309**, 107905 (2022).
18. Menger, K.: Probabilistic Geometry, Proc. Natl. Acad. Sci. USA **27**, 226-229 (1951).
19. Mishra, P.K., Sachdeva, S., Banerjee, S.K.: Some Fixed Point Theorems in b-metric Space. Turkish Journal of Analysis and Number Theory **2:1**, 19-22 (2014).
20. Parvaneh, V., Kadelburg, Z.: Extended Partial b-Metric Spaces and Some Fixed Point Results, Filomat **32**(8), 2837-2850 (2018).
21. Petre, I.-R., Bota, M.: Fixed Point Theorems on Generalized B-Metric Spaces, Publ. Math. Debrecen **83**(1-2), 139-159 (2013).
22. Prajapati, P. K. B., Bhardwaj, R.: Fixed point theorems in bi-b-metric spaces, Math. Anal. Contemp. Appl. **5**(2), 73-81 (2023).
23. Ralević, D., Paunović, I.: Removal of noise in color images and image segmentation using fuzzy metrics. Hrčak (2021).
24. Saadati, R.: On the Topology of Fuzzy Metric Type Spaces, Filomat **29**, 133-141 (2015).
25. Samphavat, S., Prinyasart, T.: On ultrametrics, b-metrics, w-distances, metric-preserving functions, and fixed point theorems. Fixed Point Theory and Algorithms for Sciences and Engineering **2024**(9) (2024).
26. Van An, T., Tuyen, L.Q., Dung, N.V.: Stone-type theorem on b-metric spaces and applications. Topology and its Applications **185-186**, 50-64 (2015).
27. Younis, M., Ahmad, A., Abdou, N.: Novel fuzzy contractions and applications to engineering science. Fractal and Fractional **8**(1), 28 (2023).
28. Zahid, M., Din, F.U., Younis, M., Ahmad, H., Öztürk, M.: Some results on multivalued proximal contractions with application to integral equation. Mathematics **12**(22), 3488 (2024).

On Application of Formal Fuzzy Contexts in Data Comparison Models

Māris Krastiņš[1] and Ingrīda Uļjane[2]

[1] University of Latvia, Department of Mathematics, Jelgavas iela 3, Riga, LV-1004, Latvia.
e-mail: mk18032@edu.lu.lv

[2] University of Latvia, Department of Mathematics, Jelgavas iela 3, Riga, LV-1004, Latvia.
e-mail: Ingrida.Uljane@lu.lv

Abstract: Noticing the problem of precise matching of fuzzy sets of objects and fuzzy sets of properties in fuzzy concept analysis and the difficulties caused by it for practical applications, we propose to replace concept lattices by more flexible, namely graded *fuzzy preconcept lattices*. Based on fuzzy preconcept lattices, we develop the foundations of the graded approach for all three versions of fuzzy concept analysis: the "classical" formal concept analysis, the object-oriented concept analysis and the property-oriented concept analysis. This has been one of the last researches carried out by professor Alexander Šostak who developed the main theoretical concepts which were further applied in practical studies by both authors.

The notion of a formal context as the basis for formal concept analysis first appeared in papers by R. Wille and B. Ganter in early 80-ties of the previous century and later developed into a full-bodied theory (see, e.g. [1] and [2]) having numerous important applications. A formal context is a triple (X, Y, R) where X, Y are non-empty sets and $R \subseteq X \times Y$ is a relation. The usual interpretation of a context (X, Y, R) is that X is the set of some abstract objects, Y is the set of some abstract properties or attributes and the entry $(x, y) \in R$ abbreviated as xRy means that an object x has the attribute y. The idea of concept analysis is to distinguish those pairs $(A, B) \in 2^X \times 2^Y$ called (formal) concepts which are in a specific way mutually connected via relation R. Specifically, according to R. Wille and B. Ganter a formal concept in a context (X, Y, R) is a pair (A, B) such that a property y is in B if and only if all $x \in A$ have property y (i.e. are related to y) and conversely, an object x is in A if and only if it has all the properties $y \in B$ (i.e. all $y \in B$ are related to x).
The concept as defined by R. Wille and B. Ganter is not the only one natural way to unite sets of objects and sets of attributes in reasonable pairs of *"concepts"* by means of the relation $R \subset X \times Y$. In particular I. Düntch and G. Gediga [3], working in the field of modal logic, suggested to consider as concepts the pairs of objects and attributes $A \subseteq X, B \subseteq Y$ such that a property y is in B if and only if all objects having these properties are contained in A and an object x is in A if and only if it has at least one property $y \in B$.
In definition of a concept in a context (X, Y, R) proposed by R. Wille and B.

Ganter, the set X of objects play the same role as the set Y of attributes: if one renames the elements of X as attributes and the elements of Y as objects the pairs (A, B) presenting concepts remain the same. In other words, the context (X, Y, R) is equivalent to the context (Y, X, R). On the other hand, in definition of a context (X, Y, R) proposed by I. Düntch and G. Gediga the roles of the set X of objects and the set Y of attributes are different: a concept (A, B) in the context (X, Y, R) becomes a concept in the context (Y, X, R^{-1}). This observation prompted Y. Y. Yao [4] to present an approach which is dual to the approach used by I. Düntch and G. Gediga for comprehension of a concept in the context (X, Y, R). It is usually referred now as object-oriented as different from attribute oriented to the interpretation of a concept in a context (X, Y, R).
To develop a fuzzy version of a formal concept analysis one naturally have to start with a fuzzy version of a formal context, that is a quadruple (X, Y, L, R) where X, Y are sets L is a complete lattice, probably with an additional algebraic structure and $R : X \times Y \to L$ is an L-fuzzy relation. The fuzzy version of the concept of R. Wille and B. Ganter was introduced by R. Belohlavek in [5] while fuzzy versions of concepts introduced by I. Düntch and G. Gediga as well as Y. Y. Yao first appeared as far as we know in [6].
In our research we replace fuzzy concepts as the focus of the study by a very general notion of a preconcept viewing it as a certain "potential concept". Further, by means of fuzzy logic tools we prescribe to each preconcept a grade, that is the measure of its "conceptuality". In the result we obtain a graded fuzzy preconcept lattice. The graded fuzzy preconcept lattices on the lines of the classical, i.e. concept analysis of R. Wille and B. Ganter were first considered in our paper [7]. In turn, graded fuzzy concepts in the spirits of oriented concept analysis were exposed in our talk at the CLA2022 conference [8]. It is the goal of this work to incorporate further to the graded versions of of all three "branches" of fuzzy concept analysis, to compare them and to illustrate the scope of their applications in data comparison models.

References

1. Wille R.: Concept lattices and conceptual knowledge systems. Computers and Mathematics with Applications (1992), 23, pp. 493–515.
2. Ganter B., Wille R.: Formal Concept Analysis: Mathematical Foundations. Springer Verlag, Berlin, 1999.
3. Düntch I., Gediga G.: Approximation operators in qualitative data analysis, in: de Swart H., Orlowska E., Schmidt G., Roubens M. (Eds.), Theory and Application of Relational Structures as Knowledge Instruments, Springer, Heidelberg, 2003, pp. 216–233.
4. Yao Y.: Concept Lattices in Rough Set Theory, in: Dick, S., Kurgan, L., Pedrycz, W, Reformat, M. (Eds.), Proceedings. 2004 Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS 2004), June 27-30, 2004, pp. 796–801.

5. Bělohlávek, R.: Concept lattices and order in fuzzy logic. Annals of Pure and Applied Logic 128 (2004), pp. 277–298.
6. Lai, H-L., Zhang, D.: Concept lattices of fuzzy context: Formal concept analysis vs. rough set theory. International Journal of Approximate Reasoning 50 (2009), pp. 695–707.
7. Šostak, A., Uļjane I., Krastiņš M.: Gradations of fuzzy preconcept lattices, AXIOMS 10(1) (2021) art. 41.
8. Krastiņš M., Šostak A., Uļjane I.: Application of graded fuzzy concept lattices in risk analysis, in Proceedings 16^{th} Joint Conference on Computational Intelligence, Cordero P., Krídlo O. (Eds.). CEUR-WP, 3308 (2022), pp. 179–186.

Solving linear equations using generalized equality

Branimir Šešelja[1] and Andreja Tepavčević[1,2]

[1] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Novi Sad 21000, Serbia
[2] Mathematical Institute SANU, Kneza Mihajlova 36, 11000 Belgrade, Serbia
e-mail: {seselja,andreja}@dmi.uns.ac.rs

Abstract: We present a method of solving linear equations over a structure with a binary operation, in the situations of missing or corrupted (unclear) data. For this purpose we use the framework of poset-valued functions, replacing the classical equality by a symmetric and transitive map from the given structure of data into the suitable poset. This generalized equality is additionally assumed to be compatible with the given binary operation. Our technique can be considered as a new way of solving linear equations if the classical technique is not applicable.

Keywords: P-sets · generalized equality · P-quasigroups · approximate solutions of equations.

Acknowledgement: This research was supported by the Science Fund of the Republic of Serbia, # Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research- AT-MATADOR

References

1. R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/Plenum Publishers, New York, 2002.
2. R. Bělohlávek, V. Vychodil, *Algebras with fuzzy equalities*, Fuzzy Sets and Systems 157 (2006) 161–201.
3. B. Budimirović, V. Budimirović, B. Šešelja, A. Tepavčević, *E-fuzzy groups*, Fuzzy Sets and Systems, Fuzzy Sets and Systems 289 (2016) 94–112.
4. B.A. Davey, H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1992.
5. M. Demirci, *Foundations of fuzzy functions and vague algebra based on many-valued equivalence relations part I: fuzzy functions and their applications, part II: vague algebraic notions, part III: constructions of vague algebraic notions and vague arithmetic operations*, Int. J. General Systems 32 (3) (2003) 123–155, 157–175, 177–201.
6. M. Demirci, *A theory of vague lattices based on many-valued equivalence relations I: general representation results,* Fuzzy Sets and Systems 151 (2005) 437–472.
7. M. Demirci, *A theory of vague lattices based on many-valued equivalence relations II: complete lattices,* Fuzzy Sets and Systems 151 (2005) 473–489.

8. E. E. Edeghagba, B. Šešelja, A. Tepavčević, *P-algebras*, Axioms 14(2) (2025) 81.
9. U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems 158 (2007) 1143–1174.
10. A. Krapež, B. Šešelja, A. Tepavčević, *Solving Linear Equations in Fuzzy Quasigroups* , Information Sciences, 491, (2019), 179–189.
11. J. Medina, V. Stepanovic, A. Tepavcevic *Solutions of matrix equations with weak fuzzy equivalence relations*, Information Sciences 629 (2023) 634–645.
12. E. Horvath, L. Kwuida. B. Šešelja, A. Tepavčević, *P-sets*, Fuzzy Sets and Systems 503 (2025) 109244.

Degree of validity of attribute implications based on multi-adjoint concept-forming operators

M. Eugenia Cornejo, Jesús Medina, and Francisco José Ocaña

Department of Mathematics, University of Cádiz, Cádiz, Spain
e-mail: {mariaeugenia.cornejo, jesus.medina, franciscojose.ocana}@uca.es

Abstract: The study of attribute implications in fuzzy settings plays a crucial role in knowledge representation and data analysis with vague or imprecise information. In this paper, we propose a new definition of validity of an attribute implication within the fuzzy framework of multi-adjoint concept lattices, taking into account the original definition in terms of the derivation operators.

Keywords: Attribute implications · Validity · Multi-adjoint concept lattices

Acknowledgement: Partially supported by the project PID2019-108991GB-I00 funded by MICIU/AEI/10.13039/501100011033, the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGeneration EU/PRTR, and by the industrial predoctoral contract PU/EPIF-FPI-GRUPOENERGETICOPUERTOREAL/CP/2022-051, corresponding to the Research and Transfer Promotion Program of the University of Cádiz 2018/2019.

References

1. L. Antoni, P. Eliač, J. Gunič, D. Kotlárová, S. Krajči, O. Krídlo, P. Sokol, and L. Šnajder. Bimorphisms and attribute implications in heterogeneous formal contexts. *International Journal of Approximate Reasoning*, 172:109245, 2024.
2. R. Belohlávek and V. Vychodil. Attribute dependencies for data with grades I,. *International Journal of General Systems*, 45(7-8):864–888, 2016.
3. M. E. Cornejo, J. Medina, and F. J. Ocaña. Attribute implications in multi-adjoint concept lattices with hedges. *Fuzzy Sets and Systems*, 479:108854, 2024.
4. M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. A comparative study of adjoint triples. *Fuzzy Sets and Systems*, 211:1–14, 2013.
5. M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Computing the validity of attribute implications in multi-adjoint concept lattices. In *International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2016*, volume II, pages 414–423, 2016.
6. M. E. Cornejo, J. Medina, E. Ramírez-Poussa, and C. Rubio-Manzano. Preferences in discrete multi-adjoint formal concept analysis. *Information Sciences*, 650:119507, 2023.

7. D. Dubois, J. Medina, and H. Prade. Extracting attribute implications from a formal context: Unifying the basic approaches. *Information Sciences*, 689:121419, 2025.
8. D. Dubois, J. Medina, H. Prade, and E. Ramírez-Poussa. Disjunctive attribute dependencies in formal concept analysis under the epistemic view of formal contexts. *Information Sciences*, 561:31–51, 2021.
9. B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundation*. Springer Verlag, 1999.
10. J.-L. Guigues and V. Duquenne. Familles minimales d'implications informatives résultant d'un tableau de donnés binaires. *Mathmatiques et Sciences Humaines*, 95:5–18, 1986.
11. V. Liñeiro-Barea, J. Medina, and I. Medina-Bulo. Generating fuzzy attribute rules via fuzzy formal concept analysis. *Studies in Computational Intelligence*, 758:105–119, 2018.
12. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.
13. F. Pérez-Gámez, P. Cordero, M. Enciso, and A. Mora. Simplification logic for the management of unknown information. *Information Sciences*, 634:505–519, 2023.

Comparison of Formal Concept Analysis and k-means Clustering on Healthcare Dataset

Štefan Puci[1], Matúš Semančík[1], Dominika Kotlárová[1], Peter Eliaš[2], Ján Guniš[1], Ľubomír Antoni[1], Stanislav Krajči[1], Ondrej Krídlo[1] and Ľubomír Šnajder[1]

[1] Institute of Computer Science, Faculty of Science Pavol Jozef Šafárik University in Košice, 041 80 Košice, Slovakia

[2] Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01, Košice, Slovak Republic

e-mail: stefan.puci@student.upjs.sk, matus.semancik@student.upjs.sk, dominika.kotlarova@student.upjs.sk, elias@saske.sk, jan.gunis@upjs.sk, lubomir.antoni@upjs.sk, stanislav.krajci@upjs.sk, ondrej.kridlo@upjs.sk, lubomir.snajder@upjs.sk

Abstract: The modified Rice-Siff algorithm [1,2] within the framework of (Fuzzy) Formal Concept Analysis [3,4,5,6] represents a method for efficient computation of formal concepts. This algorithm builds upon the foundational Rice-Siff approach and the modification incorporates optimizations to improve scalability and computational efficiency when applied to larger and more complex datasets. By modifying the formal concept generation process, the algorithm reduces redundancy and uses the structural properties of concept lattices. This enhancement is particularly relevant for applications in data analysis, knowledge representation, and other scientific domains [7,8]. Other interesting and powerful methods of fuzzification in Formal concept analysis were developed by various research groups [9,10,11,12,13].

The k-means clustering algorithm, one of the most widely used unsupervised machine learning algorithms for partitioning a dataset into distinct clusters based on similarity, was first proposed by Stuart Lloyd in 1957. Independently, the method was later investigated by James MacQueen in 1967, who denoted the term k-means and described its statistical properties in a clustering context [14]. The algorithm is based on minimizing intra-cluster variance while maximizing inter-cluster distance, providing a grouping of data points. Despite its interpretability and computational efficiency, k-Means has limitations, such as sensitivity to the initial selection of centroids and difficulty handling clusters of varying sizes or densities. It was applied in various fields, including image segmentation, customer segmentation, or bioinformatics [15].

In this paper, we compare the modified Rice-Siff algorithm and k-means clustering on the publicly available dataset collected by the consortium I-CARE Cardiac Arrest REsearch [16]. Seven hospitals across the United States and Europe participated in this study, including two from The Netherlands, one from

Belgium, and three from Boston and New Haven, USA. The dataset comprises metadata and EEG recordings from 607 patients. Each patient contributed multiple EEG files, reflecting variations in the duration of monitoring. These recordings were obtained using a 19-channel longitudinal EEG setup. We constructed a tabular dataset from the metadata and our extracted features of EEG recordings in tabular form. Thus, we applied the methods of the modified Rice-Siff algorithm and k-means clustering on the constructed dataset and compared the similarity of the obtained clusters of patients. In summary, we present the differences between both approaches (e.g., modified Rice-Siff can generate overlapping clusters) and their advantages and formulate recommendations for data analysts.

Keywords: Formal Concept Analysis · Rice-Siff algorithm · k-means clustering · Healthcare are

Acknowledgement: This work was supported by the Slovak Research and Development Agency under contract No. APVV-21-0468 (Ľubomír Antoni). This article was supported by the Scientific Grant Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic under contract VEGA 1/0539/25 (Ondrej Krídlo, Stanislav Krajči, Dominika Kotlárová, Štefan Puci, Matúš Semančík) entitled by Proposal of advanced methods in the field of Formal concept analysis and their application and by the Scientific Grant Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic and Slovak Academy of Sciences under contract VEGA 2/0097/20 (Peter Eliaš). This research was also supported by the Internal Scientific Grant System of the Faculty of Science, Pavol Jozef Šafárik University in Košice, under the project entitled Proposal of Formal Concept Analysis Methods and Their Applications to Real-World Data (Dominika Kotlárová).

References

1. Krajči, S.: Cluster-based efficient generation of fuzzy concepts. Neural Netw. World 13(5), 521-530 (2003)
2. Krajči, S.: Social network and formal concept analysis. In: Social Networks: A Framework of Computational Intelligence, Studies in Computational Intelligence, vol. 526, pp. 41-61. Springer (2013)
3. Ganter, G., Wille, R.: Formal Concept Analysis, Mathematical Foundation. Springer Verlag (1999)
4. Bělohlávek, R.: Concept lattices and order in fuzzy logic. Ann. Pure Appl. Logic **128**, 277–298 (2004)
5. Burusco, A., Fuentes-González, R.: The study of L-fuzzy concept lattice. Mathw. Soft Comput. **3**, 209–218 (1994)
6. Ben Yahia, S., Jaoua, A.: Discovering knowledge from fuzzy concept lattice. In: Kandel, A., Last, M., Bunke, H. (eds.) Data Mining and Computational Intelligence, pp. 169–190. Physica-Verlag (2001)
7. Antoni, L., Krajči, S., Krídlo, O.: Randomized fuzzy formal contexts and relevance of one-sided concepts. In: Formal Concept Analysis, Lecture Notes in Computer Science, vol. 9113, pp. 163-176. Springer (2015)

8. Antoni, L., Krajči, S., Krídlo, O.: On fuzzy generalizations of concept lattices. In: Interactions Between Computational Intelligence and Mathematics, Studies in Computational Intelligence, vol. 758, pp. 79-103. Springer (2018)
9. Medina, J., Ojeda-Aciego, M., Ruiz-Calviño, J.: Formal concept analysis via multi-adjoint concept lattices. Fuzzy Sets Syst. **160**, 130–144 (2009)
10. Cornejo, M.E., Medina, J., Ocaña, F.J.: Attribute implications in multi-adjoint concept lattices with hedges. Fuzzy Sets Syst. 479, 108854 (2024)
11. Madrid, N., Ojeda-Aciego, M.: Multi-adjoint lattices from adjoint triples with involutive negation. Fuzzy Sets Syst. **405** 88–105 (2021)
12. Pócs, J.: Note on generating fuzzy concept lattices via Galois connections. Inf. Sci. **185**, 128–136 (2012)
13. Medina-Moreno, J., Ojeda-Aciego, M., Pócs, J., Ramírez-Poussa, E.: On the Dedekind-MacNeille completion and formal concept analysis based on multilattices. Fuzzy Sets Syst. 303, 1–20 (2016)
14. MacQueen, J.: Some methods for classification and analysis of multivariate observations. In: Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, vol. 1, pp. 281-297. University of California Press (1967)
15. Xu, R., Wunsch, D.: Cluster analysis and k-means clustering: An introduction. In: Clustering, pp. 1-34. Springer, Berlin, Heidelberg (2012).
16. Goldberger, A.L., Amaral, L.A., Glass L., et al.: PhysioBank, PhysioToolkit, and PhysioNet: Components of a new research resource for complex physiologic signals. Circulation. 101(23): e215-e220 (2023)

***Keynote speech*:**

Category Theory in Action: A Novel Framework for Knowledge Graph Analysis and Reasoning

Marek Reformat

Department of Electrical and Computer Engineering,
University of Alberta, Canada

Abstract: Knowledge graphs are suitable for modeling semantically rich data, but their analytical potential is unexplored. This talk describes a formal mathematical framework that employs category theory to reason about and analyze knowledge graphs at different levels of abstraction.

Using an explanatory academic collaboration graph, we illustrate how category notions like coverings, sieves, and Grothendieck topologies can reinforce different contextual viewpoints of a knowledge graph, enabling multi-perspective reasoning about entities and their relationships. We also illustrate how sheaves and co-sheaves provide mechanisms for data analysis and its cohesion across the graph structure.

Our approach enables a novel approach to analysis, including context-dependent reasoning, multi-scale reasoning, and local-to-global formalization of information flow.

Decision-making based on confirmation measures

Fernando Chacón-Gómez, M. Eugenia Cornejo and Jesús Medina

Department of Mathematics, University of Cádiz, Cádiz, Spain
e-mail: {fernando.chacon,mariaeugenia.cornejo,jesus.medina}@uca.es

Abstract: Confirmation measures have been introduced in Rough Set Theory to analyze the influence of the antecedent on the consequent of decision rules. This paper extends these measures to the fuzzy setting and proposes their application in data classification to address decision-making problems. Furthermore, a particular confirmation measure is defined, studied in detail and applied to a real case.

Keywords: Fuzzy Rough Set Theory · Decision Rules · Confirmation Measures · Data Classification.

Acknowledgement: Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGenerationEU/PRTR, and by the Research and Transfer Promotion Program of the University of Cádiz.

References

1. R. G. Aragón, M. E. Cornejo, J. Medina, J. Moreno-García, and E. Ramírez-Poussa. Decision support system for photovoltaic fault detection avoiding meteorological conditions. *International Journal of Information Technology & Decision Making*, 21(03):911–932, 2022.
2. F. Chacón-Gómez, M. E. Cornejo, and J. Medina. Decision making in fuzzy rough set theory. *Mathematics*, 11(19), 2023.
3. F. Chacón-Gómez, M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Rough set decision algorithms for modeling with uncertainty. *Journal of Computational and Applied Mathematics*, 437:115413, 2024.
4. C. Cornelis, J. Medina, and N. Verbiest. Multi-adjoint fuzzy rough sets: Definition, properties and attribute selection. *International Journal of Approximate Reasoning*, 55:412–426, 2014.
5. S. Greco, B. Matarazzo, and R. Słowiński. Parameterized rough set model using rough membership and bayesian confirmation measures. *International Journal of Approximate Reasoning*, 49(2):285–300, 2008.
6. S. Greco, Z. Pawlak, and R. Słowiński. Can Bayesian confirmation measures be useful for rough set decision rules? *Engineering Applications of Artificial Intelligence*, 17(4):345–361, 2004.

7. Y. Lv, X. Guo, Q. Zhou, L. Qian and J. Liu. Predictive maintenance decision-making for variable faults with non-equivalent costs of fault severities. *Advanced Engineering Informatics*, 56, 2023.
8. A. Nakamura. Fuzzy rough sets. *Note on Multiple-Valued Logic in Japan*, 9:1-8, 1988.
9. Z. Pawlak. Rough sets. *International Journal of Computer and Information Science*, 11:341–356, 1982.
10. Z. Pawlak. *Rough Sets: Theoretical Aspects of Reasoning About Data*. Kluwer Academic Publishers, Norwell, MA, USA, 1992.
11. Z. Pawlak. Rough sets and decision algorithms. In W. Ziarko and Y. Yao, editors, *Rough Sets and Current Trends in Computing*, pp. 30-45, Berlin, Heidelberg, 2001.
12. G. Piatetsky-Shapiro. Discovery, analysis, and presentation of strong rules. In G. Piatetsky-Shapiro and W. J. Frawley, editors, *Knowledge Discovery in Databases*, pages 229–248. AAAI/MIT Press, 1991.
13. R. Rosati, L. Romeo, G. Cecchini et al. From knowledge-based to big data analytic model: a novel IoT and machine learning based decision support system for predictive maintenance in Industry 4.0. *Journal of Intelligent Manufacturing*, 34:107–121, 2023.
14. R. Słowiński and J. Stefanowski. Rough classification with valued closeness relation. In E. Diday, Y. Lechevallier, M. Schader, P. Bertrand, and B. Burtschy, editors, *New Approaches in Classification and Data Analysis*, pages 482–489, Berlin, Heidelberg, 1994.
15. I. Szczech. *Multicriteria Attractiveness Evaluation of Decision and Association Rules*, pages 197–274. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.
16. V. M. Vargas, R. Rosati, C. HervÃ¡s-MartÃnez, et al. A hybrid feature learning approach based on convolutional kernels for ATM fault prediction using event-log data. *Engineering Applications of Artificial Intelligence*, 123, 2023.
17. A. Wieczorek and R. Słowiński. Generating a set of association and decision rules with statistically representative support and anti-support. *Information Sciences*, 277:56–70, 2014.
18. L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

Network Visualization of a Fuzzy Measure

Carlos I. Pérez-Sechi[1], Javier Castro[1,2], Daniel Gómez[1,2], Daniel Martín[1,2], Rosa Espínola[1,2] and Inmaculada Gutiérrez [1,2]

[1] Faculty of Statistical Studies, Complutense University of Madrid, Ciudad Universitaria, Av. Puerta de Hierro, 28040, Madrid, Spain, https://estudiosestadisticos.ucm.es
e-mail: cperez14@ucm.es, jcastroc@estad.ucm.es, dagomez@estad.ucm.es, daniem05@ucm.es, rosaev@estad.ucm.es, inmaguti@ucm.es

[2] Instituto de Estadística y Ciencia de los Datos, Complutense University of Madrid, Ciudad Universitaria, Av. Puerta de Hierro, 28040, Madrid, Spain

Abstract: This conference paper proposes a visualization method of fuzzy measures to model complex interactions between elements in dynamic environments. Despite the utility of fuzzy measures, they present challenges due to the complexity of their interpretation when the number of elements rises. We suggest using graphical representations of the Shapley Index and the Murofushi-Grabisch interaction index, to provide a more intuitive understanding of fuzzy measures. This approach enables faster and smarter decisions by simplifying the analysis of elements interaction.

Keywords: Decision Modeling · Soft Computing · Fuzzy Measures · Social Networks · Graph Theory

Acknowledgement: This research has been partially supported by the Spanish Government I+D+i Plans PID2020-116884GB-I00 and PID2021-122905NB-C21.

References

1. Castro, J., Gómez, D., Molina, E., Tejada, J.: Improving polynomial estimation of the Shapley value by stratified random sampling with optimum allocation. Computers & Operations Research **82**, 180–188 (2017)
2. Castro, J., Gómez, D., Tejada, J.: Polynomial calculation of the Shapley value based on sampling. Computers & Operations Research **36**(5), 1726–1730 (2009)
3. Dubois, D.: Possibility theory and statistical reasoning. Computational Statistics & Data Analysis **51**(1), 47–69 (2006), the Fuzzy Approach to Statistical Analysis
4. Eberhard, K.: The effects of visualization on judgment and decision-making: a systematic literature review. Management Review Quarterly **73**(1), 167–214 (2023)
5. Falatoonitoosi, E., Leman, Z., Sorooshian, S., Salimi, M.: Decision-making trial and evaluation laboratory. Research Journal of Applied Sciences, Engineering and Technology **5**(13), 3476–3480 (2013)
6. Fujimoto, K.: Representations of importance and interaction of fuzzy measures, capacities, games and its extensions: A survey. In: Integrated Uncertainty Management and Applications, pp. 115–126. Springer, Berlin, Heidelberg (2010)

7. Grabisch, M.: The representation of importance and interaction of features by fuzzy measures. Pattern Recognition Letters **17**(6), 567–575 (1996)
8. Grabisch, M.: k-order additive discrete fuzzy measures and their representation. Fuzzy Sets and Systems **92**(2), 167–189 (1997)
9. Grabisch, M., Roubens, M.: An axiomatic approach to the concept of interaction among players in cooperative games. International Journal of Game Theory **28**, 547–565 (1999)
10. Gutiérrez, I., Castro, J., Gómez, D., Española, R.: Calculating the interaction index of a fuzzy measure: A polynomial approach based on sampling. Fuzzy Sets and Systems **466**, 108454 (2023)
11. Malone, D.W.: An introduction to the application of interpretive structural modeling. Proceedings of the IEEE **63**(3), 397–404 (1975)
12. Mandic, K., Bobar, V., Delibašić, B.: Modeling interactions among criteria in mcdm methods: a review. In: Decision Support Systems V–Big Data Analytics for Decision Making: First International Conference, ICDSST 2015, Belgrade, Serbia, May 27-29, 2015, Proceedings 1. pp. 98–109. Springer (2015)
13. Molnar, C.: Interpreting machine learning models with SHAP. A guide with Python examples and theory on Shapley values. `https://christophm.github.io/interpretable-ml-book/` (2024), accessed: 2025-02-13
14. Murofushi, T., Soneda, S.: Techniques for reading fuzzy measures (iii): Interaction index. In: Proceedings of the 9th Fuzzy Systems Symposium, Sapporo, Japan. pp. 693–696 (1993)
15. Saaty, T.L.: Decision Making with Dependence and Feedback: The Analytic Network Process. RWS, Pittsburgh, PA (1996)
16. Shapley, L.S.: 17. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games (AM-28), Volume II, pp. 307–318. Princeton University Press, Princeton (1953)
17. Smerecnik, C.M., Mesters, I., Kessels, L.T., Ruiter, R.A., De Vries, N.K., De Vries, H.: Understanding the positive effects of graphical risk information on comprehension: measuring attention directed to written, tabular, and graphical risk information. Risk Analysis: An International Journal **30**(9), 1387–1398 (2010)
18. Sugeno, M.: Fuzzy measure and fuzzy integral. Transactions of the Society of Instrument and Control Engineers **8**(2), 218–226 (1972)

Complex-Valued Neural Networks: Applications and Performance in Image Classification

Piotr A. Kowalski[1,2] and Michał Rutkowski[1]

[1] Faculty of Physics and Applied Computer Science, AGH University of Krakow, al. A. Mickiewicza 30, 30-059 Krakow, Poland.
e-mail: pkowal@agh.edu.pl

[2] Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland.

Abstract: This paper investigates the use of complex-valued neural networks in the image classification task, using the MNIST dataset as a case study. While real-valued neural networks have been extensively applied in various machine learning problems, complex-valued networks offer an alternative approach that can better model certain types of data, particularly in domains where oscillatory or periodic patterns are prevalent. We explore the theoretical foundations of complex-valued networks, highlighting their key differences from real-valued networks, particularly in the handling of input data and the computation of gradients during backpropagation. The study presents an experiment comparing the performance of complex-valued and real-valued networks for image classification, with the input data transformed into the complex domain using a discrete Fourier transform. The results demonstrate that complex-valued networks can achieve competitive performance, with faster convergence and higher accuracy in early training epochs, particularly when using the Adam optimization algorithm. This research provides insights into the potential advantages of complex-valued neural networks and their applicability to tasks involving complex data, suggesting new directions for future work in this area.

Keywords: Complex-Valued Neural Networks · Complex-Valued transfer function · image classification · MNIST dataset.

Acknowledgement: This work was partially supported by the program "Excellence Initiative- Research University" for the AGH University of Krakow and by a Grant for Statutory Activity from the Faculty of Physics and Applied Computer Science of the AGH University of Krakow.

References

1. Alagözlü, M.: Stochastic gradient descent variants and applications (02 2022). https://doi.org/10.13140/RG.2.2.12528.53767

2. Barrachina, J.A., Ren, C., Vieillard, G., Morisseau, C., Ovarlez, J.P.: Theory and implementation of complex-valued neural networks. arXiv preprint arXiv:2302.08286 (2023)
3. Benvenuto, N., Piazza, F.: On the complex backpropagation algorithm. IEEE Transactions on Signal Processing **40**(4), 967–969 (1992)
4. Deng, L.: The mnist database of handwritten digit images for machine learning research [best of the web]. IEEE signal processing magazine **29**(6), 141–142 (2012)
5. Hirose, A.: Complex-valued neural networks, vol. 400. Springer Science & Business Media (2012)
6. Kingma, D.P., Ba, J.: Adam: A method for stochastic optimization. arXiv preprint arXiv:1412.6980 (2014)
7. Kowalski, P.A., Kusy, M., Kubasiak, S., Łukasik, S.: Probabilistic neural network - parameters adjustment in classification task. In: 2020 International Joint Conference on Neural Networks (IJCNN). pp. 1–8 (2020). https://doi.org/10.1109/IJCNN48605.2020.9207361
8. Kowalski, P.A., Walczak, M.: Feature selection for regression tasks base on explainable artificial intelligence procedures. In: 2023 International Joint Conference on Neural Networks (IJCNN). pp. 1–8 (2023). https://doi.org/10.1109/IJCNN54540.2023.10191064
9. Leung, H., Haykin, S.: The complex backpropagation algorithm. IEEE Transactions on signal processing **39**(9), 2101–2104 (1991)
10. Mao, A., Mohri, M., Zhong, Y.: Cross-entropy loss functions: Theoretical analysis and applications. arXiv preprint arXiv:2304.07288 (2023)
11. Narkhede, M.V., Bartakke, P.P., Sutaone, M.S.: A review on weight initialization strategies for neural networks. Artificial intelligence review **55**(1), 291–322 (2022)
12. Netrapalli, P.: Stochastic gradient descent and its variants in machine learning. Journal of the Indian Institute of Science **99**(2), 201–213 (2019)
13. Nitta, T.: A theoretical foundation for the complex-valued neural network. Neural Networks **16**(5-6), 733–744 (2003). https://doi.org/10.1016/S0893-6080(03)00105-6
14. Rutkowski, L.: Metody i techniki sztucznej inteligencji. Wydawnictwo Naukowe PWN (2009)

The granular structure of covering-based rough sets generated by the Mapper algorithm

Wannes De Maeyer[1], Mauricio Restrepo[2] and Chris Cornelis[1]

[1] Computational Web Intelligence. Department of Mathematics, Computer Science and Statistics, Ghent University. Ghent, Belgium
e-mail: wannes.demaeyer@ugent.be, chris.cornelis@ugent.be

[2] Department of Mathematics. Universidad Militar Nueva Granada. Bogotá, Colombia
e-mail: mauricio.restrepo@unimilitar.edu.co

Abstract: Covering-based roughs sets generalize the notion of rough sets by changing the partition of a universe to a covering. In earlier work, the use of the Mapper algorithm (which was originally designed to visualize high dimensional datasets) to construct coverings of datasets was introduced. In this paper we combine Mapper and an adapted version of the DBSCAN clustering algorithm and prove that the generated coverings maintain the same granular structure as the lower-dimensional coverings that are used as input for Mapper. In particular, we investigate the case where the associated order between coverings is linear.

Keywords: Granular structure · Covering-based rough sets · Mapper algorithm

References

1. Pawlak, Z.: Rough sets. *International Journal of Computer and Information Sciences* **11**(5), 341–356 (1982)
2. Yao, Y., Yao, B.: Covering based rough set approximations. *Information Sciences* **200**, 91–107 (2012)
3. Zhu, P., Wen, Q.: Entropy and co-entropy of a covering approximation space. *International Journal of Approximate Reasoning* **53**, 528–540 (2012)
4. Restrepo, M., Cornelis, C., Gómez, J.: Partial order relation for approximation operators in covering based rough sets. *Information Sciences* **284**, 44–59 (2014)
5. D'eer, L., Restrepo, M., Cornelis, C., Gómez, J.: Neighborhood operators for covering-based rough sets. *Information Sciences* **336**, 21–44 (2016)
6. Ma, L.: On some types of neighborhood-related covering rough sets. *International Journal of Approximate Reasoning* **53**(6), 901–911 (2012)
7. Restrepo, M., Cornelis, C., Gómez, J.: Duality, conjugacy and adjointness of approximation operators in covering-based rough sets. *International Journal of Approximate Reasoning* **55**(1, Part 4), 469–485 (2014). Rough Sets and Logic
8. Restrepo, M., Cornelis, C.: Mapper-based rough sets. In: Hu, M., Cornelis, C., Zhang, Y., Lingras, P., Ślęzak, D., Yao, J. (eds.) *Rough Sets*, pp. 3–17. Springer, Cham (2024)

9. Yao, Y.: On generalizing rough set theory. In: *Lecture Notes in Computer Science*, vol. 2639, pp. 44–51 (2003)
10. Van Veen, H.J., Saul, N., Eargle, D., Mangham, S.W.: Kepler Mapper: A flexible Python implementation of the Mapper algorithm. Zenodo (2021)
11. Ester, M., Kriegel, H.-P., Sander, J., Xu, X.: A density-based algorithm for discovering clusters in large spatial databases with noise. In: *Proceedings of the Second International Conference on Knowledge Discovery and Data Mining*, pp. 226–231 (1996)
12. Everitt, B.S., Landau, S., Leese, M., Stahl, D.: *Cluster Analysis*, 5th edn. Wiley Series in Probability and Statistics. Wiley-Blackwell, Hoboken, NJ (2011)
13. Chazal, F., Michel, B.: An introduction to topological data analysis: Fundamental and practical aspects for data scientists. *Frontiers in Artificial Intelligence* **4** (2017)

Problem of distributivity between 2-uninorms and quasy-linear means

Dragan Jočić[1] and Ivana Štajner-Papuga[2]

[1] Mathematical Institute of the Serbian Academy of Sciences and Arts, Serbia
e-mail: jocicndragan@gmail.com

[2] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia
e-mail: ivana.stajner-papuga@dmi.uns.ac.rs

Abstract: Distributivity laws between certain classes of aggregation operators play a fundamental role in a large number of disciplines such as fuzzy sets and fuzzy logic, pseudo-analysis and measure theory, and particulary in the utility theory. This paper is focused on the problem of distributivity between some classes of 2-uninorms and quasi-linear means.

Keywords: Uninorms · 2-uninorms · quasi-linear means · distributivity equations

References

1. Aczel, J.: Lectures on functional Equations and their Applications. Academic Press (1966)
2. Akella, P.: Structure of n-uninorms. Fuzzy Sets and Systems **158**, 1631–1651 (2007)
3. Calvo, T.: On some solutions of the distributivity equations. Fuzzy Sets and Systems **104**, 85–96 (1999)
4. Drewniak, J., Drygaś, P., Rak, E.: Distributivity between uninorms and nullnorms. Fuzzy Sets and Systems **159**, 1646–1657 (2008)
5. Dubois, D., Pap, E., Prade, H.: Hybrid probabilictic-possibilistic mixtures and utility functions. Preferences and Decisions under Incomplete Knowledge, Studies in Fuzziness and Soft Computing **51**, 51–73 Springer-Verlag (2000)
6. Fodor, J.C., Yager, R.R., Rybalov, A.: Structure of uninorms. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **5**, 411–427 (1997)
7. Grabisch, M., Marichal, J., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge Uviversity Press (2009)
8. Jočić, D., Štajner-Papuga, I.: Some implications of the restricted distributivity of aggregation operators with absorbing elements for utility theory. Fuzzy Sets and Systems **291**, 54–65 (2016)
9. Jočić, D., Štajner-Papuga, I.: Distributivity laws for quasi-linear means. Iranian Journal of Fuzzy Systems **19 (1)**, 1–11 (2022)
10. Klement, E. P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer Academic Publishers, Dordrecht (2000)
11. Mesiarová-Zemánková, A.: The n-uninorms with continuous underlying t-norms and t-conorms. International Journal of General Systems **50 (1)**, 92–116 (2021)
12. Mesiarová-Zemánková, A.: Characterization of idempotent n-uninorms. Fuzzy Sets and Systems **427**, 1–22 (2022)

13. Wang, J.M., Qin, F.: On the characterization of distributivity equations about quasi-arithmetic means. Aequationes Mathematicae **90**, 501–515 (2016)
14. Zong, W., Su, Y., Liu H.-W., De Baets, B.: On the structure of 2-uninorms. Information Sciences **467**, 506–527 (2018)

Unifunctions, a new class of functions for bipolar modeling

Juraj Kalafut and Martin Kalina

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering
Slovak University of Technology
e-mail: juraj.kalafut@stuba.sk, martin.kalina@stuba.sk

Abstract: Uninorms [3], which were proposed as a common basis for t-norms and t-conorms, found their application in various areas. One of the reasons is their bipolar behavior, i.e., uninorms are strongly conjunctive below the neutral element, disjunctive above, and averaging otherwise. However, the axioms of uninorms imply that there is no uninorm that is continuous on the entire unit square, which leads to several problems that must be overcome, namely computational stability and the inability to make gradual changes.

In [1] Bustince et al. defined overlap functions as a parallel structure to t-norms. In this definition they impose continuity rather than associativity, arguing that in many cases of application associativity may be superfluous, but continuity is not. One of these cases is when the function is strictly binary and thus the n-ary form is unnecessary. Similarly, grouping functions [2] have been defined to replace t-conorms in an analogical way.

Following these steps, in this paper we want to introduce a new class of binary functions, called unifunctions, which on the one hand resemble uninorms as much as possible and on the other hand preserve continuity on the whole unit square. Since every uninorm is composed of a t-norm and a t-conorm, we want to preserve such a connection between a unifunction on the one side and an overlap and a grouping function on the other side. Therefore, the definition of a unifunction is as follows.

Definition 1. *Let* $e \in]0,1[$. *A binary function* $\mathrm{UF}_e : [0,1]^2 \to [0,1]$ *is called a unifunction if the following conditions hold.*

1. UF_e *is non-decreasing.*
2. UF_e *is symmetric.*
3. *For* $(x,y) \in [0,e]^2$, $\mathrm{UF}_e(x,y) = 0$ *if and only if* $x = 0$ *or* $y = 0$.
4. *For* $(x,y) \in [e,1]^2$, $\mathrm{UF}_e(x,y) = 1$ *if and only if* $x = 1$ *or* $y = 1$.
5. $\mathrm{UF}_e(x,y) = e$ *then* $x < e < y$ *or* $y < e < x$ *or* $x = y = e$.
6. UF_e *is continuous*

Although none of the unifunctions is associative, several interesting properties can be established, such as 1-Lipschitz, self-duality, and others. Note that a unifunction restricted to

- $[0,e]^2$ is isomorphic to an overlap function,
- $[e,1]^2$ is isomorphic to a grouping function,

– $[0, e] \times [e, 1]$ is isomorphic to a continuous bipolar function.

Acknowledgement: This contribution was supported by grants VEGA 1/0036/23, VEGA 2/0128/24, APVV SK-SRB-23-0044 and Program na podporu mladých výskumníkov (Young Researchers Support Programme).

References

1. H. Bustince, J. Fernandez, R. Mesiar, J. Montero, R. Orduna (2010). Overlap functions. Nonlinear Analysis, Theory, Methods and Applications 72 (3-4), pp. 1488–1499.
2. H. Bustince, M. Pagola, R. Mesiar, E. Hüllemeyer, F. Herrera (2012). Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. IEEE Transactions on Fuzzy Systems 20 (3), pp. 405–415.
3. R. R. Yager, A. Rybalov (1996). Uninorm aggregation operators. Fuzzy Sets and Systems 80, pp. 111–120.

***Keynote speech*:**

Normative Reasoning with Deontic Answer Set Programming

Pedro Cabalar

Computer Science and Information Technology Department, University of A Coruña, Spain

Abstract: A rigorous verification of the compliance of Artificial Intelligence (AI) systems with social, ethical and legal regulations requires, in the first place, the design of formal languages for an accurate specification of norms. Normative reasoning falls under the domain of deontic logic, that studies the formalisation of concepts such as obligations, permissions and violations. In this talk, we will review a recent approach to normative reasoning based on a deontic extension of Answer Set Programming (ASP), a successful paradigm for practical Knowledge Representation and problem solving. To this aim, we will start reviewing the extension of Equilibrium Logic (the logical basis of ASP) to cope with deontic operators and their combination with explicit negation. Then, we will illustrate the use of this approach by formalising several challenging examples from the literature, implementing them on the deontic ASP tool Â«deolingoÂ». Finally, we will move to consider temporal normative reasoning and the kind of concepts that arise when we combine deontic operators with time, explaining their recent formalisation based on Temporal Deontic Equilibrium Logic.

Formal Verification of Social Engineering in Information-Seeking Dialogues

Andreas Brännström and Juan Carlos Nieves

Umeä University, Department of Computing Science, SE-901 87, Umeä, Sweden
e-mail: {andreasb, jcnieves}@cs.umu.se

Abstract: In this paper, we apply formal dialogue methods to recognize and analyze phishing in interactions. Phishing attacks exploit human vulnerabilities, typically through deceptive and manipulative messages, leading victims to disclose sensitive information. Existing machine learning-based detection methods often lack transparency, making it difficult to trace manipulation tactics. We utilize the so-called Goal-Hiding Dialogue (GHD) framework, originally designed for reasoning about non-collaborative agents in information-seeking dialogues. The framework employs Quantitative Bipolar Argumentation Frameworks (QBAFs) to model how a seeker agent strategically influences a target's willingness to engage with certain topics. Our approach provides a mathematically grounded method for identifying key conversational shifts where manipulation occurs, contributing to phishing detection and evidence analysis.

Keywords: Formal dialogues · Quantitative argumentation · Social engineering · Phishing · Information extraction · Non-collaborative agents

References

1. Ansari, S., Alam, M.Z.: Computational approaches to persuasion detection and potential of use in social engineering. In: Proceedings of the Future Technologies Conference. pp. 394–409. Springer (2023)
2. Ayeni, R.K., Adebiyi, A.A., Okesola, J.O., Igbekele, E.: Phishing attacks and detection techniques: A systematic review. In: 2024 International Conference on Science, Engineering and Business for Driving Sustainable Development Goals (SEB4SDG). pp. 1–17. IEEE (2024)
3. Baroni, P., Rago, A., Toni, F.: From fine-grained properties to broad principles for gradual argumentation: A principled spectrum. International Journal of Approximate Reasoning **105**, 252–286 (2019)
4. Black, E., Hunter, A.: An inquiry dialogue system. Autonomous Agents and Multi-Agent Systems **19**(2), 173–209 (2009)
5. Brännström, A., Dignum, V., Nieves, J.C.: Goal-hiding information-seeking dialogues: A formal framework. International Journal of Approximate Reasoning **177**, 109325 (2025). https://doi.org/https://doi.org/10.1016/j.ijar.2024.109325, https://www.sciencedirect.com/science/article/pii/S0888613X24002123
6. Chatterjee, I., Goršič, M., Clapp, J.D., Novak, D.: Automatic estimation of interpersonal engagement during naturalistic conversation using dyadic physiological measurements. Frontiers in Neuroscience **15**, 757381 (2021)
7. He, D., Lv, X., Xu, X., Yu, S., Li, D., Chan, S., Guizani, M.: An effective double-layer detection system against social engineering attacks. IEEE Network **36**(6), 92–98 (2022)

8. Hobert, S., Følstad, A., Law, E.L.C.: Chatbots for active learning: A case of phishing email identification. International Journal of Human-Computer Studies **179**, 103108 (2023)
9. Jones, K.S., Armstrong, M.E., Tornblad, M.K., Siami Namin, A.: How social engineers use persuasion principles during vishing attacks. Information & Computer Security **29**(2), 314–331 (2021)
10. Koide, T., Chiba, D., Akiyama, M.: To get lost is to learn the way: Automatically collecting multi-step social engineering attacks on the web. In: Proceedings of the 15th ACM Asia Conference on Computer and Communications Security. pp. 394–408 (2020)
11. Lansley, M., Polatidis, N., Kapetanakis, S., Amin, K., Samakovitis, G., Petridis, M.: Seen the villains: Detecting social engineering attacks using case-based reasoning and deep learning (2019)
12. Mashtalyar, N., Ntaganzwa, U.N., Santos, T., Hakak, S., Ray, S.: Social engineering attacks: Recent advances and challenges. In: International Conference on Human-Computer Interaction. pp. 417–431. Springer (2021)
13. Mele, I., Muntean, C.I., Nardini, F.M., Perego, R., Tonellotto, N., Frieder, O.: Topic propagation in conversational search. In: Proceedings of the 43rd International ACM SIGIR conference on research and development in Information Retrieval. pp. 2057–2060 (2020)
14. Mouton, F., Leenen, L., Venter, H.S.: Social engineering attack examples, templates and scenarios. Computers & Security **59**, 186–209 (2016)
15. Ni, J., Ábrego, G.H., Constant, N., Ma, J., Hall, K.B., Cer, D., Yang, Y.: Sentence-t5: Scalable sentence encoders from pre-trained text-to-text models. arXiv preprint arXiv:2108.08877 (2021)
16. Salloum, S., Gaber, T., Vadera, S., Shaalan, K.: Phishing email detection using natural language processing techniques: a literature survey. Procedia Computer Science **189**, 19–28 (2021)
17. Salloum, S., Gaber, T., Vadera, S., Shaalan, K.: A systematic literature review on phishing email detection using natural language processing techniques. IEEE Access **10**, 65703–65727 (2022)
18. Schmitt, M., Flechais, I.: Digital deception: Generative artificial intelligence in social engineering and phishing. Artificial Intelligence Review **57**(12), 1–23 (2024)
19. Shalke, C.J., Achary, R.: Social engineering attack and scam detection using advanced natural langugae processing algorithm. In: 2022 6th International Conference on Trends in Electronics and Informatics (ICOEI). pp. 1749–1754. IEEE (2022)
20. Siddiqi, M.A., Pak, W., Siddiqi, M.A.: A study on the psychology of social engineering-based cyberattacks and existing countermeasures. Applied Sciences **12**(12), 6042 (2022)
21. Stappen, L., Baird, A., Cambria, E., Schuller, B.W.: Sentiment analysis and topic recognition in video transcriptions. IEEE Intelligent Systems **36**(2), 88–95 (2021)
22. Syafitri, W., Shukur, Z., Asma'Mokhtar, U., Sulaiman, R., Ibrahim, M.A.: Social engineering attacks prevention: A systematic literature review. IEEE access **10**, 39325–39343 (2022)
23. Team, O.: Chatgpt: Optimizing language models for dialogue (2022)
24. Thoppilan, R., De Freitas, D., Hall, J., Shazeer, N., Kulshreshtha, A., Cheng, H.T., Jin, A., Bos, T., Baker, L., Du, Y., et al.: Lamda: Language models for dialog applications. arXiv preprint arXiv:2201.08239 (2022)
25. Wang, Z., Sun, L., Zhu, H.: Defining social engineering in cybersecurity. IEEE Access **8**, 85094–85115 (2020)

Blueprint Personas in Digital Forensics and Cybersecurity

Stefania Costantini[1,3], Alina Vozna[1,2], Andrea Monaldini[1,2] and Raffaele Olivieri[1]

[1] Department of Information Engineering, Computer Science and Mathematics, University of L'Aquila, Italy
e-mail: stefania.costantini@univaq.it, alina.vozna@student.univaq.it, andrea.monaldini@student.univaq.it, raffaele.olivieri@gmail.com
[2] University of Pisa, Largo B. Pontecorvo, Pisa, 56127, Italy
[3] Gruppo Nazionale per il Calcolo Scientifico - INdAM, Rome, Italy

Abstract: As cyber threats become increasingly sophisticated, traditional security measures struggle to adapt. This paper explores the integration of Blueprint Personas, a structured approach to modeling cyber threats, into AI-driven digital forensics and cybersecurity. By simulating adversary behaviors, these personas enhance threat intelligence, adversary emulation, and forensic investigations. Additionally, we introduce the Reference Ontology of Trust (ROT) to improve explainability and adaptive decision-making in AI-driven security systems. We also propose Intelligent Agents to automate threat detection and forensic analysis within Security Operations Centers (SOCs). This work-in-progress lays the foundation for a multi-agent security framework, aiming to enhance proactive cyber defense and real-time forensic automation.

Keywords: Intelligent Agent · Healthcare · Blueprint Personas · Ontology · Trust.

Acknowledgement: Research partially supported by the PNRR Project CUP E13C 24000430006 "Enhanced Network of intelligent Agents for Building Livable Environments - ENABLE", and by PRIN 2022 CUP E53D23007850001 Project TrustPACTX - Design of the Hybrid Society Humans-Autonomous Systems: Architecture, Trustworthiness, Trust, EthiCs, and EXplainability (the case of Patient Care), and by PRIN PNNR CUP E53D23016270001 ADVISOR - ADaptiVe legIble robotS for trustwORthy health coaching.

References

1. Regulation (eu) 2016/679 of the european parliament and of the council. Official Journal of the European Union (2016), `https://eur-lex.europa.eu/eli/reg/2016/679/oj`, accessed: Feb. 19, 2025
2. Alahmadi, B.A., Axon, L., Martinovic, I.: 99% false positives: A qualitative study of {SOC} analysts' perspectives on security alarms. In: 31st USENIX Security Symposium (USENIX Security 22). pp. 2783–2800 (2022)

3. Amaral, G.C.M., Sales, T.P., Guizzardi, G., Porello, D., Guarino, N.: Towards a reference ontology of money: Monetary objects, currencies and related concepts (short paper). In: International Workshop on Value Modeling and Business Ontologies (2020), https://api.semanticscholar.org/CorpusID:213190497
4. Aminu, E.F., Oyefolahan, I.O., Abdullahi, M.B., Salaudeen, M.T.: A review on ontology development methodologies for developing ontological knowledge representation systems for various domains (2020)
5. Applebaum, A., Miller, D., Strom, B., Korban, C., Wolf, R.: Intelligent, automated red team emulation. In: Proceedings of the 32nd annual conference on computer security applications. pp. 363–373 (2016)
6. Assistance, H.C.: Summary of the hipaa privacy rule. Office for Civil Rights (2003)
7. Bahrami, P.N., Dehghantanha, A., Dargahi, T., Parizi, R.M., Choo, K.K.R., Javadi, H.H.: Cyber kill chain-based taxonomy of advanced persistent threat actors: Analogy of tactics, techniques, and procedures. Journal of information processing systems **15**(4), 865–889 (2019)
8. Baig, Z.A., Szewczyk, P., Valli, C., Rabadia, P., Hannay, P., Chernyshev, M., Johnstone, M., Kerai, P., Ibrahim, A., Sansurooah, K., et al.: Future challenges for smart cities: Cyber-security and digital forensics. Digital Investigation **22**, 3–13 (2017)
9. Bittle, C.M.: The Domain of Being: Ontology. BoD–Books on Demand (2022)
10. Commission, E.: Blueprint for digital transformation of health and care in an ageing society. https://digital-strategy.ec.europa.eu/en/library/blueprint-digital-transformation-health-and-care-ageing-society (2024), accessed: 2024-09-23
11. Dingledine, R., Mathewson, N., Syverson, P.F., et al.: Tor: The second-generation onion router. In: USENIX security symposium. vol. 4, pp. 303–320 (2004)
12. Grimm, S., Abecker, A., Völker, J., Studer, R.: Ontologies and the semantic web. In: Handbook of semantic web technologies (2011)
13. Jethava, G., Rao, U.P.: Exploring security and trust mechanisms in online social networks: An extensive review. Computers & Security p. 103790 (2024)
14. Kearney, P., Abdelsamea, M., Schmoor, X., Shah, F., Vickers, I.: Combating alert fatigue in the security operations centre. Available at SSRN 4633965 (2023)
15. Li, W., Meng, W., Kwok, L.F.: Surveying trust-based collaborative intrusion detection: state-of-the-art, challenges and future directions. IEEE Communications Surveys & Tutorials **24**(1), 280–305 (2021)
16. Monaldini, A., Vozna, A., Costantini, S.: Blueprint personas in digital health transformation (short paper). In: Calimeri, F., Dragoni, M., Stella, F. (eds.) Proceedings of the 3rd AIxIA Workshop on Artificial Intelligence For Healthcare (HC@AIxIA 2024) co-located with the 23rd International Conference of the Italian Association for Artificial Intelligence (AIxIA 2024), Bolzano, Italy, 27-28 November 2024. CEUR Workshop Proceedings, vol. 3880, pp. 40–49. CEUR-WS.org (2024)
17. Putrevu, V., Mukhopadhyay, S., Manna, S., Rani, N., Vaid, A., Chunduri, H., Putrevu, M., Shukla, S., Langehaug, T., Graham, S., et al.: Digital threats (2024)
18. Turk, E., Wontor, V., Vera-Muñoz, C., Comnes, L., Rodrigues, N., Ferrari, G., Moen, A.: Human-centered integrated care pathways for co-creating a digital, user-centric health information solution. Journal of Integrated Care **30**(4), 296–309 (2022)
19. Tyugu, E.: Command and control of cyber weapons. In: 2012 4th International Conference on Cyber Conflict (CYCON 2012). pp. 1–11. IEEE (2012)
20. Vielberth, M., Böhm, F., Fichtinger, I., Pernul, G.: Security operations center: A systematic study and open challenges. Ieee Access **8**, 227756–227779 (2020)

SM-based Semantics for Answer Set Programs Containing Conditional Literals and Arithmetic

Zachary Hansen and Yuliya Lierler

University of Nebraska Omaha, Omaha NE 68106, USA

Abstract: Answer Set Programming (ASP) is a declarative programming paradigm that has been applied within a variety of challenging and high-consequence systems such as explainable donor-patient matching and space shuttle decision support systems. ASP programs are concise, human-readable, and benefit from well-defined semantics rooted in mathematical logic – these qualities make ASP programs attractive candidates for formal verification. Providing high levels of assurance regarding program behavior is particularly crucial for safety-critical applications. This work is part of a research stream with the long-term goal of supporting rigorous verification of ASP-based systems.

Modern answer set programming solvers such as CLINGO support advanced language constructs that improve the expressivity and conciseness of logic programs. Conditional literals are one such construct. They form "subformulas" that behave as nested implications within the bodies of logic rules. Their inclusion brings the form of rules closer to the less restrictive syntax of first-order logic. Rules with conditional literals concisely express knowledge that may be difficult to otherwise encode. For instance, conditional literals are widely employed in meta-programming – Listings 4-7 in "How to build your own ASP-based system?!" by Kaminski et al. (2021). These listings define meta encodings which compute the classical and supported models of reified logic programs; these encodings rely heavily on conditional literals. Conditional literals may also make programs easier to formally verify by reducing the number of auxiliary or inessential predicates in a program.

Typically, the semantics of ASP programs with variables are defined indirectly, via a procedure called *grounding*. Grounding turns a given program with variables into a propositional one. Then, semantics are defined for the resulting propositional program. In 2011, Ferraris, Lee, and Lifschitz proposed semantics for ASP programs that bypasses grounding. They introduced the SM operator, which turns a program (or, rather, the first-order logic formula associated with the considered program) into a classical second-order formula. The Herbrand models of this formula coincide with the answer sets of the original program. In this project, we introduce grounding-free SM operator-based semantics for logic programs containing *both* conditional literals and arithmetic. This enhances our ability to reason about the behavior of programs independently of a specific grounding context. The established semantics for programs with conditional literals relies on a translation to non-classical infinitary propositional logic. We also establish the precise correspondence between the proposed and existing semantics.

Recursive Aggregates as Intensional Functions in Answer Set Programming

Jorge Fandinno and Zachary Hansen

University of Nebraska Omaha, Omaha, NE, USA.
e-mail: {jfandinno,zachhansen}@unomaha.edu

Abstract: This paper shows that the semantics of programs with aggregates implemented by the solvers `clingo` and `dlv` can be characterized as extended First-Order formulas with intensional functions in the logic of Here-and-There. Furthermore, this characterization can be used to study the strong equivalence of programs with aggregates under either semantics. We also present a transformation that reduces the task of checking strong equivalence to reasoning in classical First-Order logic, which serves as a foundation for automating this procedure.

Acknowledgement: This research is partially supported by NSF CAREER award 2338635.

References

1. Asuncion, V., Chen, Y., Zhang, Y., Zhou, Y.: Ordered completion for logic programs with aggregates. Artificial Intelligence **224**, 72–102 (2015)
2. Bartholomew, M., Lee, J.: First-order stable model semantics with intensional functions. Artificial Intelligence **273**, 56–93 (2019)
3. Bartholomew, M., Lee, J., Meng, Y.: First-order extension of the FLP stable model semantics via modified circumscription. In: Walsh, T. (ed.) Proceedings of the Twenty-second International Joint Conference on Artificial Intelligence (IJCAI'11). pp. 724–730. IJCAI/AAAI Press (2011)
4. Cabalar, P., Fandinno, J., Fariñas del Cerro, L., Pearce, D.: Functional ASP with intensional sets: Application to Gelfond-Zhang aggregates. Theory and Practice of Logic Programming **18**(3-4), 390–405 (2018). `https://doi.org/10.1017/S1471068418000169`
5. Calimeri, F., Faber, W., Gebser, M., Ianni, G., Kaminski, R., Krennwallner, T., Leone, N., Ricca, F., Schaub, T.: ASP-Core-2: Input language format (2012), `https://www.mat.unical.it/aspcomp2013/ASPStandardization`
6. Faber, W., Pfeifer, G., Leone, N.: Semantics and complexity of recursive aggregates in answer set programming. Artificial Intelligence **175**(1), 278–298 (2011)
7. Fandinno, J., Hansen, Z., Lierler, Y.: Axiomatization of aggregates in answer set programming. In: Sycara, K., Honavar, V., Spaan, M. (eds.) Proceedings of the Thirty-sixth National Conference on Artificial Intelligence (AAAI'22). pp. 5634–5641. AAAI Press (2022), `https://ojs.aaai.org/index.php/AAAI/issue/view/507`

8. Fandinno, J., Hansen, Z., Lierler, Y.: Axiomatization of non-recursive aggregates in first-order answer set programming. Journal of Artificial Intelligence Research **80**, 977–1031 (2024)
9. Ferraris, P., Lee, J., Lifschitz, V.: Stable models and circumscription. Artificial Intelligence **175**(1), 236–263 (2011)
10. Gebser, M., Harrison, A., Kaminski, R., Lifschitz, V., Schaub, T.: Abstract Gringo. Theory and Practice of Logic Programming **15**(4-5), 449–463 (2015). https://doi.org/10.1017/S1471068415000150
11. Lee, J., Lifschitz, V., Palla, R.: A reductive semantics for counting and choice in answer set programming. In: Fox, D., Gomes, C. (eds.) Proceedings of the Twenty-third National Conference on Artificial Intelligence (AAAI'08). pp. 472–479. AAAI Press (2008)
12. Lee, J., Meng, Y.: Stable models of formulas with generalized quantifiers (preliminary report). In: Dovier, A., Santos Costa, V. (eds.) Technical Communications of the Twenty-eighth International Conference on Logic Programming (ICLP'12). vol. 17, pp. 61–71. Leibniz International Proceedings in Informatics (LIPIcs) (2012)
13. Lifschitz, V.: Strong equivalence of logic programs with counting. Theory and Practice of Logic Programming **22**(4), 573–588 (2022)
14. Lifschitz, V., Pearce, D., Valverde, A.: Strongly equivalent logic programs. ACM Transactions on Computational Logic **2**(4), 526–541 (2001). https://doi.org/10.1145/383779.383783

Temporal Rules for Fraud Detection

Przemysław Andrzej Wałęga

Queen Mary University of London, University of Oxford, UK.
e-mail: p.walega@qmul.ac.uk

Abstract: Temporal logic programming provides a range of rule-based languages tailored for reasoning about temporal properties. These include various extensions of Datalog, such as Datalog1S, which introduces an additional temporal sort and a single successor operator, and DatalognS, which generalizes this with n-ary functions for the temporal sort. Other examples include TempLog, which incorporates operators from linear temporal logic (LTL), and DatalogMTL, which leverages operators from metric temporal logic (MTL). Many of these languages support non-monotonic negation, as seen in temporal extensions of answer set programming with LTL or MTL operators, the stream reasoning language LARS, and non-monotonic extensions of DatalogMTL. Their theoretical properties, such as expressive power and computational complexity, are often well-studied, and efficient reasoning algorithms have already been implemented.

Temporal aspects of financial transactions play a vital role in detecting fraudulent activities, making temporal logic programming a valuable tool for identifying suspicious behaviors. The traditional approach involves experts manually crafting temporal rules. These rules are then applied to datasets of financial transactions, allowing to flag fraudulent accounts via logical reasoning. While this method is fully interpretable and explainable, it has significant limitations. Fraud detection rules are often complex and evolve over time, making their formulation both labor-intensive and prone to human error.

We propose two approaches to address these challenges. The first approach leverages rule-learning techniques to automatically extract fraud detection rules from temporal datasets. Recent advancements in rule learning, particularly in inductive logic programming, indicate that data-driven rule extraction can be successfully applied in various real world problems. The second approach involves training black-box machine learning models for fraud detection and subsequently explaining their decisions using temporal rules. Graph neural networks (GNNs) are particularly promising for this task due to their strong connections with logic and the growing area of temporal graph learning (TGL). During the talk we will introduce our first results obtained on both learning temporal rules from data [1] and on temporal graph neural networks [2].

References

1. Wang, D., Wałęga, P.A., Grau, B.C.: MTLearn: Extracting temporal rules using Datalog rule learners. In: Proceedings of the 21st International Conference on Principles of Knowledge Representation and Reasoning, KR (2024)
2. Wałęga, P.A., Rawson, M.: Expressive power of temporal message passing. In: Proceedings of the 39th Annual AAAI Conference on Artificial Intelligence, AAAI (2025)

Unfiltered Large Language Models: Forensic and Legal Challenges of their Criminal Use on the Dark Web. Preliminary Study

Fernando Chacón-Gómez[1], Manuel Ojeda-Hernández[2] and Dévika Pérez-Medina[3]

[1] Department of Mathematics, University of Cádiz, Cádiz, Spain.
e-mail: fernando.chacon@uca.es

[2] Department of algebra, geometry and topology, Universidad de Málaga, Málaga, Spain.
e-mail: manuojeda@uma.es

[3] Department Criminal Law and Criminology, University of Barcelona, Barcelona, Spain.
e-mail: devikaperez@ub.edu

Abstract: The rapid development of large language models (LLMs) has revolutionised natural language processing and automatic text generation across all sectors. However, the emergence and spread of unfiltered or ethically unrestricted versions of these models has introduced new threats, especially in criminal ecosystems such as the Dark Web. These LLMs, devoid of alignment constraints, are being exploited to facilitate a wide range of illicit activities.

Among the most concerning tools is OnlyFake, highlighted in Europol's IOCTA 2024 report. This platform leverages generative AI to produce high-quality fake identities and documents at scale, posing a serious risk to identity verification systems and legal processes. In addition, models such as WormGPT, FraudGPT, and DarkGPT are being marketed on underground forums to automate phishing attacks, generate undetectable malware, and facilitate large-scale fraud and identity theft, even for users with limited technical expertise.

This presentation will examine the criminal use of unfiltered LLMs from two interrelated perspectives: computer forensics and criminal law. On the one hand, from a technical forensic point of view, the main tools, technical difficulties in detecting AI-generated content and identifying the use of generative models in criminal workflows will be described. From a criminal law perspective, the challenges of attributing criminal responsibility to developers, facilitators and users will be addressed, and a critical assessment will be made of whether existing criminal frameworks are sufficient to address the specificities of AI-facilitated crimes and the future challenges for criminal police investigation.

Keywords: Artificial Intelligence · Large Language Model · Digital Forensics · Dark Web · Deepfake

Acknowledgement: Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGenerationEU/PRTR, by the Research and Transfer Promotion Program of the University of Cádiz, by the State Agency of Research (AEI), the Spanish Ministry of Science, Innovation, and Universities (MCIU), the European Social Fund (FEDER) through the research projects with reference PID2021-127870OB-I00 and (MCIU/AEI/FEDER, UE) and the VALID research project (PID2022-140630NB-I00 funded by MCIN/ AEI/ 10.13039/ 501100011033) and by COST Action CA21133 Globalization, Illicit Trade, Sustainability and Security.

Time-Dependent Triangular Fuzzy Soft Set Model for Describing Traffic Dynamics

Boldizsár Tüű-Szabó[1], Ruba Almahasneh[2] and László T. Kóczy[1,2]

[1] Széchenyi István University, Department of Information Technology, Gyor, Hungary
e-mail: tuu.szabo.boldizsar@sze.hu
[2] Budapest University of Technology and Economics, Telecommunications and Media Informatics, Budapest, Hungary

Abstract: Traffic route optimization in dynamic urban environments is challenging due to uncertainties such as congestion and fluctuating conditions during rush hours. This paper introduces a novel route optimization framework utilizing Triangular Fuzzy Numbers (TFNs) to model the time-dependent membership of road segments to traffic scenarios like jam regions and rush hour periods adapts to historical data and can integrate real-time updates for robust evaluations of traffic conditions. By integrating with time-dependent Traveling Salesman Problem (TDTSP) algorithms, the framework provides practical solutions for urban traffic management, improving route efficiency while accounting for real-world uncertainties.

Keywords: Route Optimization · Triangular Fuzzy Numbers· Fuzzy Soft Sets.

References

1. Molodtsov, D.A.: Soft set theory - first results, Computers and Mathematics with Applica-tions 37, 19-31 (1999)
2. Maji, P.K., Biswas, R., Roy, A.R.: Soft set theory, Computers and Mathematics with Applications 45, 555-562 (2003)
3. Maji, P.K., Biswas, R., Roy, A.R.: Fuzzy soft sets, Journal of Fuzzy Mathematics 9, 589- 602 (2001)
4. Maji, P.K., Roy, A.R., Biswas, R.: An application of soft sets in a decision making problem Computers and Mathematics with Applications 44, 1077-1083 (2002)
5. Yang, X.B., Yu, D.J., Yang, J.Y., Wu, C.: Generalization of soft set theory: From crisp to fuzzy case. In: B.Y. Cao (Ed.), Proceeding of the second international conference on fuzzy information and engineering, May, 23-16, Guangzhou, 2007, in: Advance on Soft Compu-ting, vol. 40, pp. 345-354., Springer-Verlag, Berlin, Heidelberg (2007)
6. Xiao, Z., Xia, S., Gong, K., Li, D.: The trapezoidal fuzzy soft set and its application in MCDM. Applied Mathematical Modelling 36(12), 5844-5855 (2012)
7. Bai, J., Qin, X., Huang, L., Chen, Q.: Dynamic Decision Making of Decision-Makers? Psychological Expectations Based on Interval Triangular Fuzzy Soft Sets. Symmetry 16(3), 276. (2024)

8. Das, S., Kumari, P., & Verma, A. K.: Triangular fuzzy soft set and its application in MADM. International Journal of Computational Systems Engineering 2(2), 85-93 (2015)
9. Lathamaheswari, M., Nagarajan, D., Kavikumar, J., Broumi, S.: Triangular interval type-2 fuzzy soft set and its application. Complex & Intelligent Systems 6, 531-544 (2020)

Learning label ranking trees using evolutionary algorithms

Arturo Fernández-Mora[1], José A. Gámez[2], Juan C. Alfaro[3] and Juan Moreno-Garcia[4]

[1] Escuela de Doctorados, Universidad de Jaén, Jaén, 23071, Andalucía, España.
e-mail: afm00108@red.ujaen.es

[2] Escuela Superior de Ingeniería Informática, Universidad de Castilla-La Mancha, Albacete, 02071, Castilla-La Mancha, España.
e-mail: jose.gamez@uclm.es

[3] Escuela Superior de Ingeniería Minera y Industrial, Universidad de Castilla-La Mancha, Almadén, 13400, Castilla-La Mancha, España.
e-mail: juancarlos.alfaro@uclm.es

[4] E. Ingeniería Industrial y Aeroespacial Toledo, Universidad de Castilla-La Mancha, Toledo, 45004, Castilla-La Mancha, España.
e-mail: Juan.Moreno@uclm.es

Abstract: This work presents a study on the application of evolutionary methods for generating label ranking trees. The goal is to order a set of classes based on their relevance or preference for each instance in the dataset. Throughout the study, a model based on evolutionary algorithms is implemented, the advantages and challenges of using evolutionary algorithms in this problem are analyzed and its performance evaluated in terms of accuracy and efficiency compared to classical approaches. Furthermore, an improvement is proposed by seeding the initial population by using the tree learned by using the greedy Label Ranking Tree algorithm. The results obtained demonstrate that the application of evolutionary algorithms to address label ranking problems gives an improvement to Label Ranking Tree in most of the datasets.

Keywords: label ranking · tree induction · evolutionary computation

Acknowledgement: This work was supported by grant PID2020-112967GBC32 funded by MCIN/AEI/10.13039/501100011033 and by ERDF "A Way of Making Europe".

References

1. Loh, W. Y. (2011). Classification and regression trees. Wiley interdisciplinary reviews: data mining and knowledge discovery, 1(1), 14-23.
2. Eiben, A. E., & Smith, J. E. (2015). Introduction to evolutionary computing. Springer-Verlag Berlin Heidelberg.

3. Freitas, A. A. (2004). A critical review of multi-objective optimization in data mining: a position paper. ACM SIGKDD Explorations Newsletter, 6(2), 77-86.
4. Whigham, P. A. (1995, July). Grammatically-based genetic programming. In Proceedings of the workshop on genetic programming: from theory to real-world applications (Vol. 16, No. 3, pp. 33-41).
5. Delgado Osuna, J. A. (2023). Desarrollo y aplicación de nuevos modelos de aprendizaje automático para el estudio del cáncer colorrectal.
6. Alfaro, J. C., Aledo, J. A., & Gámez, J. A. (2021). Learning decision trees for the partial label ranking problem. International Journal of Intelligent Systems, 36(2), 890-918.
7. Cheng, W., Hühn, J., & Hüllermeier, E. (2009, June). Decision tree and instance-based learning for label ranking. In Proceedings of the 26th annual international conference on machine learning (pp. 161-168).
8. Borda, J. D. (1781). Mémoire sur les élections au scrutin. Histoire de l'Académie Royale des Sciences.
9. Kendall, M. G. (1945). The treatment of ties in ranking problems. Biometrika, 33(3), 239-251.
10. Woodward, J. R. (2003, December). GA or GP? that is not the question. In The 2003 Congress on Evolutionary Computation, 2003. CEC'03. (Vol. 2, pp. 1056-1063). IEEE.
11. Zhao, H. (2007). A multi-objective genetic programming approach to developing Pareto optimal decision trees. Decision Support Systems, 43(3), 809-826.
12. Jankowski, D., & Jackowski, K. (2014). Evolutionary algorithm for decision tree induction. In Computer Information Systems and Industrial Management: 13th IFIP TC8 International Conference, CISIM 2014, Ho Chi Minh City, Vietnam, November 5-7, 2014. Proceedings 14 (pp. 23-32). Springer Berlin Heidelberg.
13. DeLisle, R. K., & Dixon, S. L. (2004). Induction of decision trees via evolutionary programming. Journal of chemical information and computer sciences, 44(3), 862-870.
14. Cha, S. H., & Tappert, C. C. (2009). A genetic algorithm for constructing compact binary decision trees. Journal of pattern recognition research, 4(1), 1-13.
15. Kelly, M., Longjohn, R., Nottingham, K. The UCI Machine Learning Repository, https://archive.ics.uci.edu
16. Rodrigo, E. G., Alfaro, J. C., Aledo, J. A., & Gámez, J. A. (2024). Label ranking oblique trees. Knowledge-Based Systems, 296, 111882.
17. Fisher, R. A. (1938). The statistical utilization of multiple measurements. Annals of eugenics, 8(4), 376-386.
18. Fisher, R. A. (1926). "Applications of 'Student's' distribution." Metrun, 5(3), 90-104

A multi-adjoint lattice logic based on Gödel logic

M. Eugenia Cornejo[1], Francesc Esteva[2], Luis Fariñas del Cerro[3], Lluis Godo[2] and Jesús Medina[1]

[1] Department of Mathematics, University of Cádiz, Cádiz, Spain.
e-mail: {mariaeugenia.cornejo,jesus.medina}@uca.es
[2] Instituto de Investigación en Inteligencia Artificial (IIIA-CSIC), Bellaterra, Spain.
e-mail: {esteva,godo}@iiia.csic.es
[3] IRIT-CNRS, Université de Toulouse, Toulouse, France.
e-mail: luis.farinas@irit.fr

Abstract: Multi-adjoint lattice logic (MLL) was introduced as a logic focused on capturing multi-adjoint algebras, which are general and flexible algebraic structures used, for example, as truth-values set in different formal tools to model data sets, such as, formal concept analysis, rough sets, and fuzzy relation equations. This paper enriches this logic by expanding it with two extra connectives, one associated with the Gödel implication and the one associated with the Baaz-Monteiro projection connective. As a consequence, the implication in MLL representing the ordering in the lattice becomes definable from these two operators, and vice versa.

Keywords: Bounded poset · multi-adjoint algebra · multi-adjoint lattice logic · fuzzy logic · Gödel fuzzy logic· Baaz-Monteiro operator

Acknowledgement: Partially supported by the projects PID2022-137620NB-I00, PID2022-139835NB-C21, and by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and FEDER, UE, and European Union NextGeneration EU/PRTR.

References

1. M. Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. Lecture Notes in Logic 6, 23-33, Springer, 1996.
2. R. Cignoli, F. Esteva, L. Godo, and A. Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106–112, 2000.
3. P. Cintula and C. Noguera. *Logic and Implication. An Introduction to the General Algebraic Study of Non-classical Logics*, Vol. 57 of *Trends in Logic*, Springer, 2021.
4. M. E. Cornejo, L. Fariñas del Cerro, and J. Medina. Multi-adjoint lattice logic and truth-stressing hedges. *Fuzzy Sets and Systems*, 445:43–65, 2022.

5. M. E. Cornejo, L. Fariñas del Cerro, and J. Medina. Multi-adjoint lattice logic. properties and query answering. In G. Marreiros et al. editors, *Progress in Artificial Intelligence*, pages 701–712, Cham, 2022. Springer International Publishing.
6. M. E. Cornejo, J. Medina, E. Ramírez-Poussa, and C. Rubio-Manzano. Multi-adjoint concept lattices, preferences and Bousi prolog. In V. Flores et al. editors, *Rough Sets*, pages 331–341, Cham, 2016. Springer International Publishing.
7. C. Cornelis, J. Medina, and N. Verbiest. Multi-adjoint fuzzy rough sets: Definition, properties and attribute selection. *International Journal of Approximate Reasoning*, 55:412–426, 2014.
8. J. C. Díaz-Moreno and J. Medina. Multi-adjoint relation equations: Definition, properties and solutions using concept lattices. *Information Sciences*, 253:100–109, 2013.
9. Digital forensics: Evidence analysis via intelligent systems and practices DigForASP – Action COST CA17124. `https://digforasp.uca.es`.
10. P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Press, 1998.
11. P. Hájek. On very true. *Fuzzy Sets and Systems*, 124(3):329 – 333, 2001.
12. J. Medina. Multi-adjoint property-oriented and object-oriented concept lattices. *Information Sciences*, 190:95–106, 2012.
13. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.
14. J. Medina, M. Ojeda-Aciego, and P. Vojtáš. Similarity-based unification: a multi-adjoint approach. *Fuzzy Sets and Systems*, 146:43–62, 2004.

Relations Among Intermediate Syllogisms

Petra Murinová and Vilém Novák

University of Ostrava Institute for Research and Applications of Fuzzy Modeling 30. dubna 22, 701 03 Ostrava 1, Czech Republic.

e-mail: {Petra.Murinova,Vilem.Novak}@osu.cz

Abstract: In this article, we will show how relations forming the graded Peterson's square of opposition (contrary, contradictory, sub-contrary and sub-alterns) are connected with intermediate logical syllogisms, i.e., syllogisms that contain intermediate quantifiers. Our results are related to the theory of non-monotonic reasoning.

Acknowledgement: This article has been prepared with the financial support of the project "Research of Excellence on Digital Technologies and Wellbeing CZ.02.01.01/00/22-008/0004583" that is co-financed by the European Union.

Additional support has been provided by the European Union under the REFRESH "Research Excellence For REgion Sustainability and High-tech Industries" project number CZ.10.03.01/00/22-003/0000048 via the Operational Programme Just Transition.

References

1. Antoniou, G.: Nonmonotonic Reasoning. MIT Press, Cambridge, Massachusetts (1997)
2. Béziau, J., Gan-Krzywoszyńska, K.: Handbook of abstracts of the 2nd World Congress on the Square of Opposition, Corte, Corsica, June 17-20 (2010)
3. Béziau, J., Gan-Krzywoszyńska, K.: Handbook of abstracts of the 3rd World Congress on the Square of Opposition, Beirut, Lebanon, June 26-30 (2010)
4. Béziau, J., Gan-Krzywoszyńska, K.: Handbook of abstracts of the 4th World Congress on the Square of Opposition, Roma, Vatican, May 5-9 pp. 26–30 (2014)
5. Bochman, A.: A Logical Theory of Nonmonotonic Inference and Belief Change. Springer, Berlin (2001)
6. Ciucci, D., Dubois, D., Prade, H.: Oppositions in rough set theory. In: T. Li, H.S. Nguyen, G. Wang, J.W. Grzymala-Busse, R. Janicki, A.E. Hassanien, H. Yu (eds.) Proc. 7th Int. Conf. on Rough Sets and Knowledge Technology (RSKT12) , Chengdu, Aug. 17-20, LNCS **7414**, 504–513 (2012)
7. Ciucci, D., Dubois, D., Prade, H.: The structure of oppositions in rough set theory and formal concept analysis - toward a new bridge between the two settings. In: C. Beierle, C. Meghini (eds.) Proc. 8th Int. Symp. on Foundations of Information and Knowledge Systems (FoIKS14), Bordeaux, Mar. 3-7, LNCS **8367**, 154–173 (2014)
8. Ciucci, D., Dubois, D., Prade, H.: Structures of opposition induced by relations: The boolean and the gradual cases. Annals of Mathematics and Artificial Intelligence pp. DOI: 10.1007/s10472–015–9480–8 (2016)

9. Dubois, D., Prade, H.: Gradual structures of oppositions. In: F. Esteva, L. Magdalena, J.L. Verdegay (eds.) Enric Trillas: Passion for Fuzzy Sets, Studies in Fuzziness and Soft Computing **322**, 79–91 (2015)
10. Dubois, D., Prade, H.: From blanché's hexagonal organization of concepts to formal concept analysis and possibility theory. Logica Universalis **6**, 149–169 (2012)
11. Elavsky, S., Blahošová, J., Lebedíková, M., Tkaczyk, M., Tancoš, M., Plhák, J., Sotolář, O., Šmahel, D.: Researching the links between smartphone behavior and adolescent wellbeing: The study protocol for the FUTURE (WP4) ecological momentary assessment study. JMIR Research Protocols (2022)
12. Murinová, P., Boffa, S.: From graded aristotle's hexagon to graded peterson's hexagon of opposition in fuzzy natural logic. Fuzzy Logic and Technology, and Aggregation Operators pp. 369–380 (2023)
13. Murinová, P., Novák, V.: A formal theory of generalized intermediate syllogisms. Fuzzy Sets and Systems **186**, 47–80 (2012)
14. Murinová, P., Novák, V.: Analysis of generalized square of opposition with intermediate quantifiers. Fuzzy Sets and Systems **242**, 89–113 (2014)
15. Murinová, P., Novák, V.: The structure of generalized intermediate syllogisms. Fuzzy Sets and Systems **247**, 18–37 (2014)
16. Murinová, P., Novák, V.: The theory of intermediate quantifiers in fuzzy natural logic revisited and the model of "Many". Fuzzy Sets and Systems **388**, 56–89 (2020)
17. Novák, V.: On fuzzy type theory. Fuzzy Sets and Systems **149**, 235–273 (2005)
18. Novák, V., Dvořák, A.: Formalization of commonsense reasoning in fuzzy logic in broader sense. Journal of Applied and Computational Mathematics **10**, 106–121 (2011)
19. Peterson, P.: Intermediate Quantifiers. Logic, linguistics, and Aristotelian semantics. Ashgate, Aldershot (2000)
20. Reiter, R.: Nonmonotonic reasoning. Ann. Rev. Comput. Sci. **2**, 147–186 (1987)

On testing hypotheses in fuzzy surroundings

Aleksandar Takači[1], Zagorka Lozanov-Crvenković[2], Ivana Štajner-Papuga[2], Gabrijela Grujić[3], Tatjana Došenović[1] and Dragan Jočić[4]

[1] Faculty of Technology, University of Novi Sad, Serbia.
e-mail: sasatakaci@gmail.com, tatjanad@uns.ac.rs

[2] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia.
e-mail: {zlc, ivana.stajner-papuga}@dmi.uns.ac.rs

[3] Faculty of Technical Sciences, University of Novi Sad, Novi Sad, Serbia.
e-mail: gabrijela.grujic@uns.ac.rs

[4] Mathematical Institute of the Serbian Academy of Sciences and Arts, Serbia.
e-mail: jocicndragan@gmail.com

Abstract: Various statistical models have found their place in practical applications; however, working with fuzzy-valued data still presents a challenge. This paper provides an overview of several well-known concepts of statistical hypothesis testing in fuzzy environments and introduces a new approach based on the comparison of triangular fuzzy numbers along the horizontal axis, i.e., on a horizontal fuzzy relation. The proposed method allows both the registered value of the test statistic and the reference critical value to be in fuzzy form and defines fuzzy degree of acceptance of hypotheses.

Keywords: Fuzzy sets · fuzzy numbers · fuzzy relations · comparing fuzzy values · hypotheses testing

Acknowledgement: This research was supported by the Science Fund of the Republic of Serbia, Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research-AT-MATADOR, and of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia.

References

1. Bede, B.: Fuzzy Arithmetic. In: Mathematics of Fuzzy Sets and Fuzzy Logic. Studies in Fuzziness and Soft Computing, vol. 295, Springer, Berlin, Heidelberg (2013).
2. Bodenhofer, U.: A similarity-based generalization of fuzzy orderings (PhD thesis). Johannes Kepler University, Linz (1998).
3. Carlsson, C., Fuller, R.: On possibilistic mean value and variance of fuzzy numbers. Fuzzy Sets Syst. 122, 315–326 (2001).
4. Denoeux, T.: Maximum likelihood estimation from fuzzy data using the EM algorithm. Fuzzy Sets Syst. 183, 72–91 (2011).

5. Dubois, D., Prade, H.: Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York (1980).
6. Dubois, D., Prade, H.: The mean value of a fuzzy number. Fuzzy Sets Syst. 24, 279–300 (1987).
7. Feng, Y.: Gaussian fuzzy random variables. Fuzzy Sets Syst. 111, 325–330 (2000).
8. Grujić, G., Lozanov-Crvenković, Z., Štajner-Papuga, I.: General fuzzy integral as a base for estimation of fuzzy quantities. Fuzzy Sets Syst. 326, 69–80 (2017).
9. Klement, E.P., Mesiar, R.: On the Expected Value of Fuzzy Events. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 23, 57–74 (2015).
10. Klir, G.J., Yuan, B.: Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, Upper Saddle River (1995).
11. Parchami, A., Taheri, S.M., Mashinchi, M.: Fuzzy p-value in testing fuzzy hypotheses with crisp data. Stat. Pap. 51, 209 (2010).
12. Puri, M.L., Ralescu, D.A.: Fuzzy random variables. J. Math. Anal. Appl. 114, 409–422 (1986).
13. Reche, F., Morales, M., Salmerón, A.: Statistical Parameters Based on Fuzzy Measures. Mathematics 8(11), 2015 (2020). https://doi.org/10.3390/math8112015
14. Saha, S.K., Samanta, S.K., Mondal, S.K.: A systematic review on generalized fuzzy numbers and its applications: Past, present, and future. Arch. Comput. Methods Eng. 29, 2657–2680 (2022).
15. Takači, A.: Applications of aggregation operators in fuzzy systems (Master's thesis). University of Novi Sad (2003).
16. Takači, A., Štajner-Papuga, I., Lozanov-Crvenković, Z., Jočić, D., Grujić, G., Došenović, T.: On Horizontal Fuzzy Relations and Hypotheses Testing. Acta Polytech. Hung. 21, 153–166 (2024).
17. Wu, H.C.: Statistical hypotheses testing for fuzzy data. Inf. Sci. 279, 446–459 (2005).
18. Yoshida, Y., Negoita, M.Ch., Howlett, R.J., Jain, L.C.: A mean estimation of fuzzy numbers by evaluation measures. In: Knowledge-Based Intelligent Information and Engineering Systems, Part II, LNAI 3214, Springer, pp. 1222–1229 (2004).
19. Zadeh, L.A.: Fuzzy sets. Inf. Control 8, 338–353 (1965).
20. Zadeh, L.A.: Probability measures of fuzzy events. J. Math. Anal. Appl. 23, 421–427 (1968).
21. Zadeh, L.A.: Fuzzy probabilities. Inf. Process. Manag. 19, 3148–153 (1984).

Kolmogorov Representation Theorem Explained in Terms of Fuzzy Partitions

Irina Perfilieva

University of Ostrava Institute for Research and Applications of Fuzzy Modeling 30. dubna 22, 701 03 Ostrava 1, Czech Republic and Cracow University of Technology, Faculty of Computer Science and Telecommunications Krakow, Poland.

e-mail: irina.perfilieva@osu.cz

Abstract: The main goal of this contribution is to show connection between the classical Kolmogorov-Arnold representation theorem and theoretical foundations of fuzzy modeling. Especially, we show that the class of inner functions in the Kolmogorov representation of a continuous multivariable function belongs, up to multiplicative constants, to the class of membership functions that constitute a fuzzy partition with the Ruspini condition.

Acknowledgement: This work was developed within the project of the University of Ostrava: Social Dimension of New Technologies in the Energy Sector in the Ostrava Metropolitan Area (reg. number CZ.02.01.01/00/23-021/000859), with financial support from the European Union through the Jan Amos Komenský Operational Programme.

References

1. Arnold, V.: On functions of three variables (in russian). Dokl. Akad. Nauk SSSR **114**, 679–681 (1957)
2. Cristian, J.V.R., Blanco, L., Pereira, R., Caus, M.: Kolmogorov-Arnold Networks (KANs) for Time Series Analysis. arXiv: 2405.08790 (2024)
3. Cybenko, G.: Approximation by superpositions of a sigmoidal function. Math. Control Signals Systems **2/4**, 303–314 (1989)
4. Holčapek, M., Perfilieva, I., Novák, V., Kreinovich, V.: Necessary and sufficient conditions for generalized uniform fuzzy partitions. Fuzzy Sets and Systems **277**, 97–121 (2015)
5. Holčapek, M., T. Tichý, A.: A smoothing filter based on fuzzy transform. Fuzzy Sets and Systems **180**, 69–97 (2011)
6. Kolmogorov, A.N.: On the representation of continuous functions of many variables by superposition of continuous functions of one variable and addition (in russian). Dokl. Akad. Nauk SSSR **114**, 953–956 (1957)
7. Liu, Z., Wang, Y., Vaidya, S., Ruehle, F., Halverson, J., Soljačić, M., Hou, T., Tegmark, M.: KAN: Kolmogorov-Arnold Networks. arXiv: 2404.19756 (2024)
8. Morris, S.A.: Hilbert 13: Are there any genuine continuous multivariate real-valued functions? Bulletin of the American Mathematical Society (New Series) **58/1**, 107–118 (2021)

9. Perfilieva, I.: Fuzzy transforms: theory and applications. Fuzzy Sets and Systems **157**, 993–1023 (2006)
10. Perfilieva, I.: F-transform. In: Kacprzyk, J., Pedrycz, W. (eds.) Handbook of Computational Intelligence, pp. 113–130. Springer, San Diego (2015)
11. Perfilieva, I., Holčapek, M., Kreinovich, V.: A new reconstruction from the F-transform components. Fuzzy Sets and Systems **288**, 3–25 (2016)
12. Poluektov, M., Polar, A.: A new iterative method for construction of the Kolmogorov-Arnold representation. arXiv: 2305.08194 (2023)
13. Stefanini, L.: F-transform with parametric generalized fuzzy partitions. Fuzzy Sets and Systems **180**, 98–120 (2011)

Systems of fuzzy relational equations in case of a complete codomain lattice

Vanja Stepanović[1] and Andreja Tepavčević[2,3]

[1] University of Belgrade, Faculty of Agriculture, Nemanjina 6, 11080 Beograd-Zemun, Serbia.
e-mail: vanja@agrif.bg.ac.rs
[2] Mathematical Institute of the Serbian Academy of Science and Art, Kneza Mihailova 36, 11000 Beograd, Serbia.
[3] University of Novi Sad, Faculty of Sciences, Trg Dositeja Obradovića 3, 21102 Novi Sad, Serbia.
e-mail: andreja@dmi.uns.ac.rs

Abstract: The paper deals with fuzzy relational equations in the case where fuzzy relations are defined as arbitrary mappings from the square of a domain set to a lattice. That lattice is usually a sort of special, residual, Heyting, Browerian, continuous etc. We consider the case of a complete codomain lattice, which is more general. The completeness of the codomain lattice implies the completeness of the lattice of all fuzzy relations on a set and of all its powers. Using Tarski fixed point theorem for an operator on a complete lattice, we prove the existence of a solution for a wide class of fuzzy relational equations and their systems. By doing this we generalize our previous results. Tarski theorem also implies the existence of the greatest and the least solution for the mentioned class of equations and their systems, and the fact that the sets of their solutions are complete lattices.

Keywords: Complete lattice · Fuzzy relational equations · Solvability · Extremal solutions

Acknowledgement: This research was supported by the Science Fund of the Republic of Serbia, Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research - AT-MATADOR. The authors also gratefully acknowledge the financial support of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia (Grants No. 451-03-137/2025-03/200116, 451-03-137/2025-03/200125 & 451-03-136/2025-03/200125 & 451-03-136/2025-03/200029).

References

1. Bede, B., Mathematics of Fuzzy Sets and Fuzzy Logic. Studies in Fuzziness and Soft Computing, vol 295. Springer, Berlin, Heidelberg (2013).
2. Brown, J.G.: A note on fuzzy sets. Information and Control **18**, 32–39 (1971).

3. Cornejo, M.E., Lobo, D., Medina, J., De Baets, B.: Bipolar equations on complete distributive symmetric residuated lattices: The case of a join-irreducible right-hand side. Fuzzy Sets and Systems **442**, 92-108 (2022).
4. Davey B.A., Priestley H.A.: Introduction to Lattices and Order. 2nd ed. Cambridge University Press (2002).
5. Dubois, D., Prade, H.: Fuzzy Sets and Systems - Theory and Applications. Academic Press, New York-London-Toronto (1980).
6. Goguen, J.A.: L-Fuzzy Sets. Journal of Mathematical Analysis and Applications **18**, 145–174 (1967).
7. Ignjatović, J., Ćirić, M., Šešelja, B., Tepavčević, A.: Fuzzy relational inequalities and equalities, fuzzy quasi-orders, closures and openings of fuzzy sets. Fuzzy Sets and Systems **260**, 1–24 (2015).
8. Sanchez, E.: Resolution of composite fuzzy relation equations. Information and Control **30** (1), 38–48 (1976).
9. Stepanović, V., Tepavčević, A.: Fuzzy sets (in)equations with a complete codomain lattice. Kybernetika **58** 145–162 (2022).
10. Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics **5** 285–309 (1955).
11. Zadeh, L.A.: Fuzzy sets. Information and Control **8**, 338–353 (1965).

WAMs based on a pair of n-tuples $(\mathbf{w}, \mathbf{b})$ and their relationship to OWA and IOWA operators

Andrea Stupňanová[1], Radko Mesiar[1,2], Radomír Halaš[2] and Jozef Pócs[2]

[1] Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Radlinského 11, 810 05 Bratislava, Slovakia.
e-mail: andrea.stupnanova@stuba.sk

[2] Palacký University Olomouc, Faculty of Science, 17. listopadu 12, 771 46 Olomouc, Czech Republic.

Abstract: The OWA and IOWA operators are both based on a given weighting vector $\mathbf{w} \in [0,1]^n$ and an ordering method expressed by appropriate permutations $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. We shall denote Ω as the set of all permutations. In the case of the OWA, the weighting vector is determined by the input n-tuple $\mathbf{x}$ itself. The OWA operator can be introduced by the following equation:

$$\begin{aligned} \mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = & \quad \mathrm{AM}\left(\sum_{i=1}^{n} w_i x_{\sigma(i)} \,|\, \sigma \in \Omega,\, x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}\right) \\ = & \quad \mathrm{AM}\left(\sum_{i=1}^{n} w_{\sigma^{-1}(i)}\, x_i \,|\, \sigma \in \Omega,\, x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}\right), \end{aligned}$$

where AM is the arithmetic mean. In other words, the OWA operator is simply a weighted arithmetic mean, $\mathrm{OWA}_{\mathbf{w}}(\mathbf{x}) = \mathrm{WAM}_{\mathbf{w_x}}(\mathbf{x})$, where $\mathbf{w_x} = \mathrm{AM}\left(\mathbf{w}_{\sigma^{-1}} | \, \sigma \in \Omega, x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}\right)$.
In the case of IOWA, an order-introducing n-tuple $\mathbf{u}$, which is fixed and equal for all inputs $\mathbf{x}$, is considered. Similarly, it we be defined as $\mathrm{IOWA}_{\mathbf{w},\mathbf{u}}(\mathbf{x}) = \mathrm{WAM}_{\mathbf{w_u}}(\mathbf{x})$, where $\mathbf{w_u} = \mathrm{AM}\left(\mathbf{w}_{\sigma^{-1}} | \, \sigma \in \Omega, u_{\sigma(1)} \geq \cdots \geq u_{\sigma(n)}\right)$.

Let us consider an n-tuple $\mathbf{x} \in [0,1]^n$. Our approach draws inspiration from previous representations. For any weighting vector $\mathbf{w}$ and an n-ary vector $\mathbf{b}(\mathbf{x}) \in [0,1]^n$, the pair $(\mathbf{w}, \mathbf{b}(\mathbf{x}))$ can be used to define a new weighting vector $\mathbf{w_{b(x)}} = \mathrm{AM}\left(\mathbf{w}_{\sigma^{-1}} | \, \sigma \in \Omega, b(\mathbf{x})_{\sigma(1)} \geq \cdots \geq b(\mathbf{x})_{\sigma(n)}\right)$.
In general, for any $\mathbf{x} \in [0,1]^n$, an n-tuple $\mathbf{b}(\mathbf{x}) \in [0,1]^n$ can be defined, and new generalisations of WAMs, OWAs and IOWAs are introduced. To demonstrate the proposed approach, several examples are presented, including related figures.

Keywords: IOWA operator · OWA operator · Weighted arithmetical mean

Acknowledgement: This study was funded by Slovak VEGA grant 1/0036/23.

References

1. Beliakov, G., James, S.: Induced Ordered Weighted Averaging Operators. In: Yager, R.R., Kacprzyk, J., Beliakov, G. (eds) Recent Developments in the OWA Operators. Studies in Fuzziness and Soft Computing 265, 29–47, Springer, Berlin, Heidelberg (2011)
2. Beliakov, G., Pradera, A., Calvo, T.: Aggregation Functions: a Guide for Practitioners. Springer, 2007.
3. Halaš, R., Mesiar, R., Pócs, J., Stupňanová, A., Jin, L.: A note on weighting vectors as a basis of particular aggregation functions. submitted paper
4. Merigó, J.M., Gil-Lafuente, A.M.: The induced generalized OWA operator. Information Sciences 179, 729–741 (2009)
5. Torra, V.: The weighted OWA operator. Int. J. of Intel. Syst. 12, 153—166 (1997)
6. Yager, R.R.: On ordered weighted averaging aggregation operators in multicriteria decisionmaking. IEEE Transaction Systems, Man and Cybernetics 18(1), 183–190 (1988).

***Keynote speech*:**

Entropies, measures of uncertain and imprecise information

Bernadette Bouchon-Meunier[1,2]

[1] National Centre for Scientific Research and Sorbonne Université, Paris, France
[2] Department of Databases and Machine Learning in the Computer Science Laboratory of the University Pierre et Marie Curie-Paris 6 (LIP6), Paris, France

Abstract: After the introduction of entropies in a probabilistic framework, this concept has been extended to the evaluation of the fuzziness or non-probabilistic uncertainty to evaluate the imprecision or imperfection associated with the observation of objects or events, for instance in the case of intuitionistic or interval-valued fuzzy sets. We focus on the common properties and the differences between such quantities, mainly the monotonicity with respect to a refinement of the information on these objects or events, considering diverse views of this refinement, subsuming or weakening the classic properties of additivity and recursivity. The utilisation of such entropies in artificial intelligence is discussed.

Analytical Properties of General Unary Hypotheses Automaton Quantifiers

Martina Daňková

University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic.
e-mail: martina.dankova@osu.cz

Abstract: In this paper, we explore the analytical properties of general unary hypotheses automaton quantifiers. Specifically, we present two distinct representations of implicational quantifiers and a method for generalizing these representations to double implicational and equivalence quantifiers. Furthermore, we examine selected properties of implicational quantifiers in relation to fuzzy set operations and their associated properties.

Keywords: Fuzzy Quantifiers · Implicational Quantifiers · GUHA method · Fuzzy Logic · Association Rules

Acknowledgement: This research was supported by the Czech Science Foundation project No. 23-06280S.

References

1. Agrawal, R., Imieliński, T., Swami, A.: Mining association rules between sets of items in large databases. Proceedings of the ACM SIGMOD International Conference on Management of Data pp. 207–216 (1993). https://doi.org/10.1145/170035.170072
2. Dubois, D., Hüllermeier, E., Prade, H.: A systematic approach to the assessment of fuzzy association rules. Data Mining and Knowledge Discovery **13**(2), 167–192 (2006). https://doi.org/10.1007/s10618-006-0051-3
3. Hájek, P., Havránek, T.: Mechanizing Hypothesis Formation Mathematical: Foundations for a General Theory. Springer-Verlag (1978)
4. Hájek, P., Holeňa, M., Rauch, J.: The GUHA method and its meaning for data mining. Journal of Computer and System Sciences **76**(1), 34–48 (2010). https://doi.org/10.1016/j.jcss.2009.05.004, special Issue on Intelligent Data Analysis
5. Holeňa, M.: Fuzzy hypotheses for GUHA implications. Fuzzy Sets and Systems **98**(1), 101–125 (1998). https://doi.org/10.1016/S0165-0114(96)00369-7
6. Ivánek, J.: Construction of implicational quantifiers from fuzzy implications. Fuzzy Sets and Systems **151**(2), 381–391 (2005). https://doi.org/10.1016/j.fss.2004.07.002
7. Kupka, J., Rusnok, P.: Fuzzy four-fold tables: Their properties and use in fuzzy association analysis. International Journal of Approximate Reasoning **108**, 89–106 (2019). https://doi.org/10.1016/j.ijar.2019.03.002

Linear spline copulas

Radko Mesiar, Andrea Stupňanová and Fateme Kouchakinejad

Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Radlinského 11, 810 05 Bratislava, Slovakia.

e-mail: radko.mesiar@stuba.sk

Abstract: One of the most important classes of 2-dimensional aggregation functions are 2-copulas (copulas, in short). Recall that a function: $C : [0,1]^2 \to [0,1]$ is a copula whenever it has a neutral element $e = 1$, an annihilator $a = 0$, and, for each rectangle $R = [x_1, x_2] \times [y_1, y_2] \subseteq [0,1]^2$ the C-volume $V_C(R) = C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1)$ is non-negative. Note that each copula C is increasing (and thus an aggregation function) and 1-Lipschitz. For more details we recommend books [1,2,3]. Linear splines play an important role in the approximation of real functions. This fact inspired us to study and discuss copulas, which are linear splines. Observe that a linear spline copula C is affine on simplices $S_1 \ldots, S_k$ covering the unit square $[0,1]^2$ and they are overlapping on segments, possibly. Each such copula is singular with support on some finitely many segments with masses uniformly distributed over those segments. Note that segment support does not generally have to lead to a linear spline copula. As an example, consider a singular copula C given by

$$C(x,y) = \begin{cases} \min(x^2, y^2) & \text{if } x + y \leq 1 \\ \max(x^2, y^2) - |x - y| & \text{else.} \end{cases}$$

The support of this copula is the same as the support of linear spline copulas $\lambda M + (1 - \lambda)W$, $\lambda \in]0,1[$. We will discuss constructions of the linear spline copulas (e.g., Archimedean case, different types of ordinal sums, shuffles, convex combinations, etc.). Several examples will be given, including illustrative figures. Finally, observe that linear spline copulas are dense in the class of all copulas.

Keywords: Aggregation function · Copula · Linear spline.

Acknowledgement: This study was funded by Slovak VEGA grant 1/0036/23 and the third author was supported also by SAIA fund.

References

1. Durante, F., Sempi, C.: Principles of Copula Theory. CRC Press, New York (2016)
2. Joe, H.: Multivariate Models and Dependence Concepts. Chapman & Hall, London (1997)
3. Nelsen, R.B.: An Introduction to Copulas. 2nd edn. Springer Science & Business Media, New York (2006)

Some notes on permutation-equivariant decomposition integrals

Adam Šeliga

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Bratislava, Slovakia.

e-mail: adam.seliga@stuba.sk

Abstract: Non-linear integration is a powerful tool in mathematics with applications in multicriteria decision-making and image processing. While symmetric aggregation functions simplify computations [2], they often fail to capture real-world complexities. To address this, we propose a symmetry framework based on permutations, called permutation-equivariant functions, within the framework of aggregation functions to better reflect relationships between criteria or values.

Since decomposition integrals [1] are aggregation functions, we can examine the property of permutation-equivariantness in their framework. Decomposition integrals are built upon decomposition systems $\mathcal{H}$ which are non-empty sets of collections, where a collection is any non-empty subset of $2^X \setminus \{\emptyset\}$, where $X = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. For a decomposition system $\mathcal{H}$ and a monotone measure μ, i.e., a grounded non-decreasing set function on 2^X, we define the decomposition integral by

$$\mathsf{dec}^{\mu}_{\mathcal{H}}(\mathbf{x}) = \bigvee_{\mathcal{D}\in\mathcal{H}} \bigvee \Big\{ \sum_{A\in\mathcal{D}} \alpha_A \mu(A) \colon \sum_{A\in\mathcal{D}} \alpha_A \mathbf{1}_A \leq \mathbf{x}, \alpha_A \geq 0 \text{ for } A \in \mathcal{D} \Big\},$$

for all inputs $\mathbf{x} \in [0, \infty[^n$, where $\mathbf{1}_A$ denotes the indicator vector of the set $A \subseteq X$.

In the work, we establish conditions on decomposition systems and monotone measures under which the decomposition integrals are permutation-equivariant aggregation functions. Furthermore, potential applications of such integrals are proposed and explored in the context of image processing, particularly addressing the problem of edge detection.

Keywords: permutation-equivariant functions · aggregation functions · decomposition integrals

Acknowledgement: Adam Šeliga was funded by the EU NextGenerationEU through the Recovery and Resilience Plan for Slovakia under the project No. 09I03-03-V04-00276.

References

1. Even, Y., Lehrer, E.: Decomposition-integral: unifying Choquet and the concave integrals. Economic Theory 56, (2014), pp. 33-58. doi:10.1007/s00199-013-0780-0
2. Grabisch, M., Marichal, J.-L., Mesiar, R., Pap, E.: Aggregation Functions. Encyclopedia of Mathematics and Its Applications, vol. 127. Cambridge University Press, 2009. ISBN 978-0-521-51926-7.

Vertical sum of concept lattices via bonds

Roberto G. Aragón, Jesús Medina and Samuel Molina-Ruiz

Department of Mathematics. University of Cádiz, Spain.
e-mail: {roberto.aragon,jesus.medina,samuel.molina}@uca.es

Abstract: Formal concept analysis provides a significant procedure to aggregate (relational) datasets based on bonds, without altering the original information. This paper studies this procedure for aggregating fuzzy formal contexts, with a view to obtaining a vertical sum of the corresponding concept lattices.

Keywords: Formal concept analysis · bonds · multi-adjoint framework · vertical sum

Acknowledgement: Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGenerationEU/PRTR, and by the project PR2023-009 funded by the University of Cádiz.

References

1. R. G. Aragón, J. Medina, and S. Molina-Ruiz. The notion of bond in the multi-adjoint concept lattice framework. In *Advances in Artificial Intelligence*, pages 243–253, Cham, 2024. Springer Nature Switzerland.
2. R. G. Aragón, J. Medina, and E. Ramírez-Poussa. Impact of local congruences in variable selection from datasets. *Journal of Computational and Applied Mathematics*, page 113416, 2022.
3. R. G. Aragón, J. Medina, and E. Ramírez-Poussa. Context of a local congruence concept reduction. *Studies in Computational Intelligence*, 1040:103–110, 2023.
4. R. Belohlavek and M. Trneckova. Factorization of matrices with grades with overcovering. *IEEE Transactions on Fuzzy Systems*, 32(4):1641–1652, April 2024.
5. M. J. Benítez-Caballero, J. Medina, E. Ramírez-Poussa, and D. Ślęzak. Rough-set-driven approach for attribute reduction in fuzzy formal concept analysis. *Fuzzy Sets and Systems*, 391:117–138, 2020.
6. M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Multi-adjoint algebras versus non-commutative residuated structures. *International Journal of Approximate Reasoning*, 66:119–138, 2015.
7. B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2 edition, 2002.
8. B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundations*. Springer, 2 edition, 2024.
9. J. Konecny and O. Krídlo. On biconcepts in formal fuzzy concept analysis. *Information Sciences*, 375:16–29, 2017.

10. J. Konecny and M. Ojeda-Aciego. On homogeneous l-bonds and heterogeneous l-bonds. *International Journal of General Systems*, 45(2):160–186, 2016.
11. O. Krídlo, S. Krajči, and M. Ojeda-Aciego. L-bonds vs extents of direct products of two L-fuzzy contexts. In *Proceedings of Concept Lattices and Applications (CLA)*, pages 70–79, 2010.
12. O. Krídlo, D. López-Rodríguez, L. Antoni, P. Eliaš, S. Krajči, and M. Ojeda-Aciego. Connecting concept lattices with bonds induced by external information. *Information Sciences*, 648:119498, 2023.
13. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

Bipolar max-product fuzzy relation equations with floor negation

Antonio Castejón, David Lobo and Pablo López-Molina

Department of Mathematics, University of Cádiz. Spain
e-mail: {antonio.castejon,david.lobo,pablo.lopezmolina}@uca.es

Abstract: This paper introduces the concept of floor negation, which is a modification of a negation operator so that it only applies below a certain threshold and takes the value 0 above that threshold. Then, bipolar max-product fuzzy relation equations with floor negation are studied. In particular, we present a simple characterization of their solvability and we characterize the existence of maximal and minimal solutions.

Keywords: Bipolar fuzzy relation equation · Max-product composition · Involutive negation

References

1. Alcalde, C., Burusco, A., Díaz-Moreno, J.C., Medina, J.: Fuzzy concept lattices and fuzzy relation equations in the retrieval processing of images and signals. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **25**(Supplement-1), 99–120 (2017).
2. Bělohlávek, R.: Fuzzy Relational Systems: Foundations and Principles. Kluwer Academic Publishers (2002).
3. Cornejo, M.E., Lobo, D., Medina, J.: Bipolar fuzzy relation equations systems based on the product t-norm. Mathematical Methods in the Applied Sciences **42**(17), 5779–5793 (2019).
4. Cornejo, M.E., Lobo, D., Medina, J.: On the solvability of bipolar max-product fuzzy relation equations with the product negation. Journal of Computational and Applied Mathematics **354**, 520–532 (2019).
5. Cornejo, M.E., Lobo, D., Medina, J., De Baets, B.: Bipolar equations on complete distributive symmetric residuated lattices: The case of a join-irreducible right-hand side. Fuzzy Sets and Systems **442**, 92–108 (2022).
6. Cornejo, M.E., Lobo, D., Medina, J.: Optimization of partially monotonic functions subject to bipolar fuzzy relation equations. Information Sciences **648**, 119497 (2023).
7. Di Nola, A., Sanchez, E., Pedrycz, W., Sessa, S.: Fuzzy relation equations and their applications to knowledge engineering. Kluwer Academic Publishers, Norwell, MA, USA (1989).
8. Dubois, D., Prade, H.: Fuzzy relation equations and causal reasoning. Fuzzy Sets and Systems **75**(2), 119–134 (1995).
9. Freson, S., De Baets, B., De Meyer, H.: Linear optimization with bipolar max–min constraints. Information Sciences **234**, 3–15 (2013).

10. Lobo, D., Cornejo, M.E., Medina, J.: Abductive reasoning in normal residuated logic programming via bipolar max-product fuzzy relation equations. In: 2019 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (EUSFLAT 2019). Atlantis Studies in Uncertainty Modelling, vol. 1, pp. 588–594. Atlantis Press (2019).
11. Lobo, D., López-Marchante, V., Medina, J.: On the impact of sup-compositions in the resolution of multi-adjoint relation equations. Mathematical Methods in the Applied Sciences **46**(14), 15581–15598 (2023).
12. Lobo, D., Medina, J., Merkl, T.C., Pichler, R.: Minimal solutions of fuzzy relation equations via maximal independent elements. Information Sciences **690**, 121558 (2025).
13. Sanchez, E.: Resolution of composite fuzzy relation equations. Information and Control **30**(1), 38–48 (1976).

New Simplification Rules for Databases with Positive and Negative Attributes

Domingo López-Rodríguez[1], Manuel Ojeda-Hernández[2] and Carlos Bejines[1]

[1] Depto. de Matemática Aplicada, Universidad de Málaga, Andalucía Tech, Málaga, 29071, Spain.
e-mail: {dominlopez,cbejines}@uma.es

[2] Department of Computing Science, Umeå University, 901 87 Umeå, Sweden.
e-mail: manueloh@cs.umu.se

Abstract: This keywork summarises the findings in the recently published paper "D. López-Rodríguez, M. Ojeda-Hernández, and C. Bejines. *New Simplification Rules for Databases with Positive and Negative Attributes.* In: Mathematics 13.2 (2025)". This paper builds upon prior work to propose new logical equivalences for managing mixed implications in databases, introducing an iterative method for simplification that delivers concise and efficient results.

Formal Concept Analysis (FCA) revolves around the concept of a formal context, denoted as $\mathbb{K} = (G, M, I)$, where G is a set of objects, M a set of attributes, and $I \subseteq G \times M$ the incidence relation specifying which objects have which attributes. In this work, a *closed-world* assumption is adopted, meaning that if $(g, m) \notin I$, it is explicitly known that object g does not possess attribute m. Mixed attributes are introduced by considering both M, the set of positive attributes, and $\overline{M} = \{\overline{m} \mid m \in M\}$, the set of negative attributes, where $\overline{m}$ indicates the absence of m. Derivation operators $^{\uparrow}\colon 2^G \to 2^{M \cup \overline{M}}$ and $^{\downarrow}\colon 2^{M \cup \overline{M}} \to 2^G$ are defined as $X^{\uparrow} = \{m \in M \mid gIm \text{ for all } g \in X\} \cup \{\overline{m} \in \overline{M} \mid g \not I m \text{ for all } g \in X\}$ and $Y^{\downarrow} = \{g \in G \mid gIm \text{ for all } m \in Y\} \cap \{g \in G \mid g \not I m \text{ for all } \overline{m} \in Y\}$.

Mixed implications, under this point of view, are expressions $A \to B$ where $A, B \subseteq M \cup \overline{M}$. An implication $A \to B$ is valid in $\mathbb{K}$ if $B \subseteq A^{\downarrow\uparrow}$. The Simplification Logic for Mixed Attributes [1] provides axioms and inference rules to reduce redundancy in these implications. Main rules include Reflexivity ($A \to A$), Augmentation ($A \to B \vdash AC \to BC$), Simplification ($A \to B, C \to D \vdash A(C \setminus B) \to D$), Key ($A \to b \vdash A\bar{b} \to M \cup \overline{M}$) and Reduction ($Ab \to C, A\bar{b} \to C \vdash A \to C$). These rules enable the construction of simplified and efficient implicational systems that preserve the information encoded in the formal context while reducing computational complexity.

In [3], the authors presented a collection of equivalence rules to simplify mixed implications, in pursue of less redundant implicational systems. The present work provides an additional set of equivalence rules which enables a more concise representation of the implicit knowledge.

Theorem 1 ([2]). *Let $\mathbb{K} = (G, M, I)$ be a formal context and let $A, B, C, D \subseteq M\overline{M}$.*

- *If there exists* $x \in A \cap \overline{C}$, $y \in B \cap \overline{C}$ *with* $A \setminus x \subseteq C \setminus \{\overline{x}\overline{y}\}$. *Then,*
$$\{A \to B, C \to D\} \equiv \{A \to B, C \setminus \overline{x} \to D\}.$$
- *If there exist* $x \in A \cap C$, $b \in B$ *and* $d \in D$ *such that* $(A \setminus x)\overline{b} \subseteq (C \setminus x)\overline{d}$, *then*
$$\{A \to B, C \to D\} \equiv \{A \to B, C \to D \setminus d\}.$$
- *If there exist* $x \in A$, $y \in B$ *such that* $(A \setminus x)\overline{y} \subseteq C$, *then*
$$\{A \to B, C \to D\} \equiv \{A \to B, C \setminus \overline{x} \to D \setminus \overline{x}\}.$$
- *If there exist* $x \in A$, $y \in B$ *such that* $(A \setminus x)\overline{y} \subseteq CD$, *then*
$$\{A \to B, C \to D\} \equiv \{A \to B, C \to D \setminus \overline{x}\}.$$

Note that, in all cases, the total number of attributes in the implicational system is reduced as these rules are applied. Thus, their iterative application defines an algorithm for the simplification of mixed implicational systems. The effectiveness of said algorithm is compared to the one that utilizes the rules presented in [3]. Results show that the proposed algorithm [2] achieves the most concise systems compared to previous approaches. Due to the trade-off between simplification and computational cost, the result of applying only one and two iterations is also compared to the computation of the whole simplification algorithm.

Acknowledgement: This research is partially supported by the Kempe foundation, the State Agency of Research (AEI), the Spanish Ministry of Science, Innovation, and Universities (MCIU) and the European Social Fund (FEDER) through the research projects with reference JCSMK24-0053 (Kempe), PID2021-127870OB-I00 and (MCIU/AEI/FEDER, UE) and the VALID research project (PID2022-140630NB-I00 funded by MCIN/ AEI/ 10.13039/ 501100011033).

References

1. Cordero, P., Enciso, M., Mora, A., Rodriguez-Jiménez, J.M.: Inference of Mixed Information in Formal Concept Analysis, pp. 81–87. Springer International Publishing, Cham (2019)
2. López-Rodríguez, D., Ojeda-Hernández, M., Bejines, C.: New simplification rules for databases with positive and negative attributes. Mathematics **13**(2) (2025). https://doi.org/10.3390/math13020309, https://www.mdpi.com/2227-7390/13/2/309
3. Pérez-Gámez, F., López-Rodríguez, D., Cordero, P., Mora, A., Ojeda-Aciego, M.: Simplifying implications with positive and negative attributes: A logic-based approach. Mathematics **10**(4) (2022). https://doi.org/10.3390/math10040607

Formal concept analysis and its fuzzy extensions for Light-Bot game solutions

Dominika Kotlárová[1], Peter Eliaš[2], Ján Guniš[1], Ľubomír Antoni[1], Stanislav Krajči[1], Ondrej Krídlo[1] and Ľubomír Šnajder[1]

[1] Institute of Computer Science, Faculty of Science Pavol Jozef Šafárik University in Košice, 041 80 Košice, Slovakia.

[2] Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01, Košice, Slovak Republic.

e-mail: dominika.kotlarova@student.upjs.sk, jan.gunis@upjs.sk, elias@saske.sk, lubomir.antoni@upjs.sk, stanislav.krajci@upjs.sk, ondrej.kridlo@upjs.sk, lubomir.snajder@upjs.sk

Abstract: Formal Concept Analysis is a set of methods rooted in mathematical lattice theory which organize data based on objects and their common attributes [1]. Variants of Formal Concept Analysis, including fuzzy extensions, offer promising methods to analyze, visualize, and categorize student solutions. These methods help group solutions into overlapping biclusters (formal concepts) and reveal relationships between attributes [2,3,4,5,6,7,8,9,10,11,12,13]. Recently, we proposed extensions working with formal contexts of heterogeneous data structures [14,15,16,17].

Computational thinking can be understood as a collection of skills involving cognitive or thought processes, such as logical reasoning, used to solve problems and gain a deeper understanding of procedures, either with or without the use of a computer [18,19].

In [20], we presented the solution implemented to help educators explore connections between features of students' solution to problem-solving and computational thinking. The framework included tools for visualization and analysis, enabling the identification of underlying patterns in student solutions. These insights can assist teachers in predicting specific student behaviors, addressing potential errors, and proactively intervening to address misconceptions or improve learning outcomes. Using a dataset of 64 student solutions, we constructed formal contexts based on extracted attributes. FCA was employed to create concept lattices from binary contexts and derive both standard and fuzzy attribute implications to uncover attribute dependencies.

In this paper, we will present extended results on whether Formal Concept Analysis can effectively describe and analyze the diverse solutions and attribute relationships in the educational and computational thinking game Light-Bot.

The analysis highlights several key findings. First, the concept lattice for binary contexts identifies the largest biclusters, representing groups of similar solutions. Second, attribute implications provide insights into patterns, such as solutions with a higher number of executed commands. Finally, fuzzy attribute

implications reveal characteristics of solutions with inefficiencies, such as unnecessary commands, moves outside the game area, or the use of indirect recursion. These results offer a detailed understanding of student behavior and solution strategies in computational problem-solving tasks.

Keywords: Formal Concept Analysis · Fuzzy extensions · Ligh-bot game · Computational thinking

Acknowledgement: This work was supported by the Slovak Research and Development Agency under contract No. APVV-21-0468 (Ľubomír Antoni). This article was supported by the Scientific Grant Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic under contract VEGA 1/0539/25 (Ondrej Krídlo, Stanislav Krajči, Dominika Kotlárová) entitled by Proposal of advanced methods in the field of Formal concept analysis and their application and by the Scientific Grant Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic and Slovak Academy of Sciences under contract VEGA 2/0097/20 (Peter Eliaš). This research was also supported by the Internal Scientific Grant System of the Faculty of Science, Pavol Jozef Šafárik University in Košice, under the project entitled Proposal of Formal Concept Analysis Methods and Their Applications to Real-World Data (Dominika Kotlárová).

References

1. Ganter, G., Wille, R.: Formal Concept Analysis, Mathematical Foundation. Springer Verlag (1999)
2. Bělohlávek, R.: Concept lattices and order in fuzzy logic. Ann. Pure Appl. Logic **128**, 277–298 (2004)
3. Burusco, A., Fuentes-González, R.: The study of L-fuzzy concept lattice. Mathw. Soft Comput. **3**, 209–218 (1994)
4. Ben Yahia, S., Jaoua, A.: Discovering knowledge from fuzzy concept lattice. In: Kandel, A., Last, M., Bunke, H. (eds.) Data Mining and Computational Intelligence, pp. 169–190. Physica-Verlag (2001)
5. Medina, J., Ojeda-Aciego, M., Vojtáš, P.: Similarity-based unification: a multi-adjoint approach. Fuzzy Sets Syst. **146**, 43–62 (2004)
6. Cornejo, M.E., Medina, J., Ocaña, F.J.: Attribute implications in multi-adjoint concept lattices with hedges. Fuzzy Sets Syst. 479, 108854 (2024)
7. Cornejo, M. E., Medina, J., Ramírez, E.: Characterizing reducts in multi-adjoint concept lattices. Inf. Sci. **422**, 364–376 (2018)
8. Madrid, N., Ojeda-Aciego, M.: Multi-adjoint lattices from adjoint triples with involutive negation. Fuzzy Sets Syst. **405** 88–105 (2021)
9. Medina, J., Ojeda-Aciego, M.: Multi-adjoint t-concept lattices. Inf. Sci. **180**, 712–725 (2010)
10. Medina, J., Ojeda-Aciego, M.: On multi-adjoint concept lattices based on heterogeneous conjunctors. Fuzzy Sets Syst. **208**, 95–110 (2012)
11. Medina, J., Ojeda-Aciego, M., Ruiz-Calviño, J.: Formal concept analysis via multi-adjoint concept lattices. Fuzzy Sets Syst. **160**, 130–144 (2009)

12. Pócs, J.: Note on generating fuzzy concept lattices via Galois connections. Inf. Sci. **185**, 128–136 (2012)
13. Medina-Moreno, J., Ojeda-Aciego, M., Pócs, J., Ramírez-Poussa, E.: On the Dedekind-MacNeille completion and formal concept analysis based on multilattices. Fuzzy Sets Syst. 303, 1–20 (2016)
14. Krajči, S.: A generalized concept lattice, Logic J. IGPL **13**, 543–550 (2005)
15. Krídlo, O., Krajči, S., Antoni, Ľ.: Formal Concept Analysis of higher order. Int. J. Gen. Syst. **45**(2), 116–134 (2016)
16. Antoni, Ľ., Krajči, S., Krídlo, O., Macek, B., Pisková, L.: On heterogeneous formal contexts. Fuzzy Sets Syst. **234**, 22–33 (2014)
17. Antoni, Ľ., Eliaš, P., Krajči, S., Krídlo, O.: Heterogeneous formal context and its decomposition by heterogeneous fuzzy subsets. Fuzzy Sets Syst. **451**, 361–384 (2022)
18. Csizmadia, A., Curzon, P., Dorling, M., Humphreys, S., Ng, T., Selby, C., Woollard, J.: Computational Thinking: A Guide for Teachers. Computing at School (2015)
19. Schindler, M., Lilienthal, A.J.: Students' Creative Process in Mathematics: Insights from Eye-Tracking-Stimulated Recall Interview on Students' Work on Multiple Solution Tasks. Int. J. Sci. Math. Educ. 18, 1565–1586 (2020)
20. Guniš, J., Šnajder, L., Antoni, L., Eliaš, P., Krídlo, O., Krajči, S.: Formal Concept Analysis of Students' Solutions on Computational Thinking Game. IEEE Transactions on Education. https://doi.org/10.1109/TE.2024.3442612 (2024)

Composition as a fuzzy conjunction between indexes of inclusion

Nicolás Madrid[1] and Manuel Ojeda-Aciego[2]

[1] Depto. de Matemáticas. Universidad de Cádiz. Spain.
e-mail: nicolas.madrid@uma.es
[2] Depto. Matemática Aplicada. Universidad de Málaga. Spain.
e-mail: aciego@uma.es

Abstract: The φ-index of inclusion was presented originally in [6] as a novel approach to model the inclusion between fuzzy sets and it was extended to general L-fuzzy sets in [3]. The main difference with respect to the existing approaches in the literature is that the inclusion between fuzzy sets is represented by mappings, instead of by values in the lattice of truth degrees.

Since this seminal approach [6], many different results have been obtained to support its use as a good generalization in the fuzzy setting of the classical inclusion of sets. For example, the φ-index of inclusion is in accordance with most of the pivotal axiomatic approaches of inclusion measures between fuzzy sets [2]; the φ-index of inclusion and the φ-index of contradiction [1] can be used to define a square of opposition, in the line of the Aristotelian one [5]; the mapping that defines φ-index of inclusion can be used in a modus ponnens inference which is optimal with respect to the set of modus ponens defined from residuated pairs [4].

In this talk we go one step further in the research of the φ-index of inclusion showing that the composition of functions can be used as a suitable conjunction in the set of indexes of inclusion. It is worth to point out two very interesting results. The first one. Let $\mathcal{G}$ be the set of mappings $f\colon [0,1] \to [0,1]$ satisfying that $f(x) \leq x$ and there exists $g\colon [0,1] \to [0,1]$ such that (f,g) is an adjoint pair. Then, the composition is a $\mathcal{G}$-fuzzy conjunction associated to two residuated implications denoted $\swarrow_{\circ}$ and $\nearrow^{\circ}$; i.e., $(\mathcal{G}, \circ, \swarrow_{\circ}, \nearrow^{\circ}, id)$ is a birresiduated lattice.

The second result is a representation theorem obtained for biresiduated lattices (which includes the standard residuated lattices) in $[0,1]$.

Theorem 2. *Let* $([0,1], *, \nearrow, \searrow, 1)$ *a biresiduated lattice, then there exists* $\Theta \subseteq \mathcal{G}$ *such that* $([0,1], *, \nearrow, \searrow, 1)$ *is isomorphic to* $(\Theta, \circ, \swarrow_{\circ}, \nearrow^{\circ}, id)$.

In other words, we can identify each biresiduated lattice in $[0,1]$ with a sub-biresiduated lattice of $(\mathcal{G}, \circ, \swarrow_{\circ}, \nearrow^{\circ}, id)$. In particular, any t-norm can be identified with the composition $\circ$ in a subset of $\mathcal{G}$.

Keywords: Galois Connections · Inclusion measure · Residuated lattices

Acknowledgement: Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant

TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGenerationEU/PRTR.

References

1. H. Bustince, N. Madrid, and M. Ojeda-Aciego. The notion of weak-contradiction: definition and measures. *IEEE Transactions on Fuzzy Systems*, 23:1057–1069, 2015.
2. N. Madrid and M. Ojeda-Aciego. A view of f-indexes of inclusion under different axiomatic definitions of fuzzy inclusion. *Lecture Notes in Artificial Intelligence*, 10564:307–318, 2017.
3. N. Madrid and M. Ojeda-Aciego. Functional degrees of inclusion and similarity between L-fuzzy sets. *Fuzzy Sets and Systems*, 390:1–22, 2020.
4. N. Madrid and M. Ojeda-Aciego. The f-index of inclusion as optimal adjoint pair for fuzzy modus ponens. *Fuzzy Sets and Systems*, 466:108474, 2023.
5. N. Madrid and M. Ojeda-Aciego. Approaching the square of opposition in terms of the f-indexes of inclusion and contradiction. *Fuzzy Sets and Systems*, 476:108769, 2024.
6. N. Madrid, M. Ojeda-Aciego, and I. Perfilieva. f-inclusion indexes between fuzzy sets. In *Proc. of IFSA-EUSFLAT*. Proc. of IFSA-EUSFLAT, 2015.

Towards the use of the φ-index of inclusion for fuzzy reasoning

Carolina Díaz-Montarroso, Nicolás Madrid and Eloísa Ramírez-Poussa

Depto. de Matemáticas. Universidad de Cádiz. Spain.
e-mail: nicolas.madrid@uma.es, eloisa.ramirez.@uca.es

Abstract: In this talk we present a novel Fuzzy Inference System (FIS) based on the f-inclusion [3]. The f-inclusion is a generalization of the notion of inclusion in Fuzzy Set Theory (FST) in accordance with the pivotal axiomatic approach of Sinha-Dougherty [2]. The main difference with respect to other approaches dealing with inclusions in FST is that the inclusion is represented by functions instead of by values in $[0, 1]$. This functional representation allows to model a Generalized Modus Ponens based on the f-inclusion [1], which is the the core of the inference engine of the proposed FIS.

The proposed FIS is supported on a formal semantics where the notions of models and logical consequence can be formally defined. As a result, the FIS can be considered a formal fuzzy logic systems, which provides reliability and theoretical grounding. The obtained FIS can be compared with the so-called Relational FIS, but there are two main differences:

- The RFIS require a fuzzy relation to link the universes of X and Y and to compute the output. In our approach, the connection between universes is left to interpretations and models. This simplifies the computation of the output in our FIS.
- The RFIS require to fix a pair of operators in advance; in general an implication operator and a conjunction operator. However, we do not need to fix operators in advance, the inference is done directly by a mapping that represents the inclusion between fuzzy sets.

Under our opinion, these two differences make of approach more accesible to the applied audience.

Keywords: Galois Connections · Inclusion measure · φ-index of inclusion · Fuzzy sets · Relational Fuzzy Inference Systems

Acknowledgement: Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, by the grant TED2021-129748B-I00 funded by MCIN/AEI/10.13039/501100011033 and European Union NextGenerationEU/PRTR.

References

1. C. Díaz-Montarroso, N. Madrid, and E. Ramírez-Poussa. Towards a Generalized Modus Ponens Based on the f-Index of Inclusion. In *Conceptual Knowledge Structures: First International Joint Conference, CONCEPTS 2024, Cádiz, Spain, September 9–13, 2024, Proceedings*, pages 36–48, Berlin, Heidelberg, 2024. Springer-Verlag. event-place: Cádiz, Spain.
2. N. Madrid and M. Ojeda-Aciego. Functional degrees of inclusion and similarity between L-fuzzy sets. *Fuzzy Sets and Systems*, 390:1–22, July 2020.
3. N. Madrid, M. Ojeda-Aciego, and I. Perfiljeva. f-inclusion indexes between fuzzy sets. In *Proceedings of the 2015 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology*, pages 1528–1533, 01 2015.

Author Index